Macroeconomics 1 (5/7)

The growth model with product variety (Romer, 1990)

Olivier Loisel

ENSAE

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Key features of the model

- Romer's (1990) model explains long-term growth by **a voluntary and remunerated technical progress**, the remuneration taking the form of a monopoly rent protected by a trade secret or a patent.

- It modelizes technological progress as **the expansion of product variety** – hence the name of "growth model with (expanding) product variety."

- The constant returns of product variety will
  - generate long-term growth,
  - imply no conditional convergence.

- Imperfect competition, which is the source of the remuneration of technological progress, will give a role to economic policy.

Literature

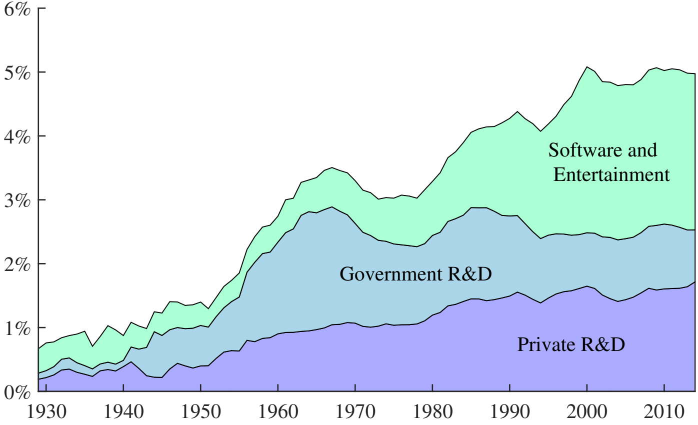
- In growth models with product variety, the goods whose variety expands can be
  - **production goods**, as in Romer (1990),
  - **consumption goods**, as in Grossman and Helpman (1991, ch. 3).

- In the original version of Romer's (1990) model, the cost of research and development ($\equiv$ R&D, necessary for inventing new types of goods) is specified in terms of *labor*.

- In this chapter, we consider a simplified version of Romer's (1990) model, in which the R&D cost is specified in terms of *goods*, as in Rivera-Batiz and Romer (1991).

R&D expenditures in the US, 1929-2013

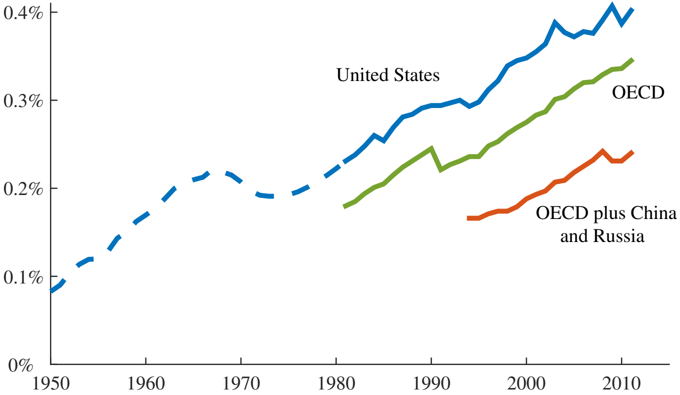
(in % of GDP)



Source: Jones (2015).

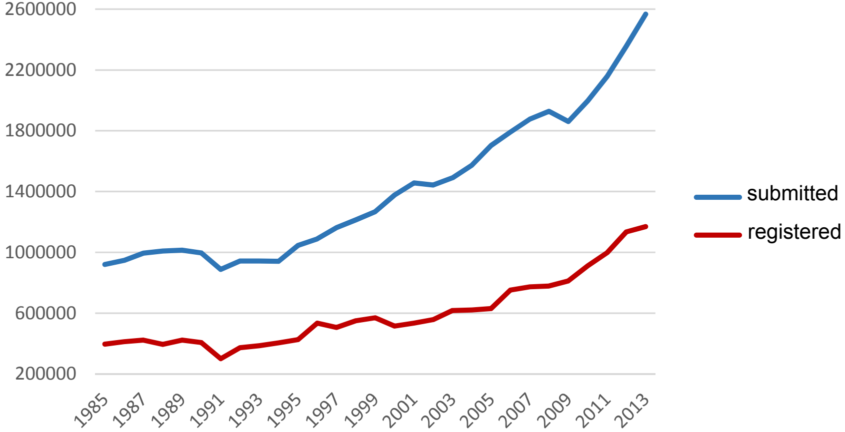
Share of researchers in the population...

...in different countries or groups of countries, 1950-2011



Source: Jones (2015).

Number of new patents in the world, 1985-2013



Source: World Intellectual Property Organization.

General overview of the model I

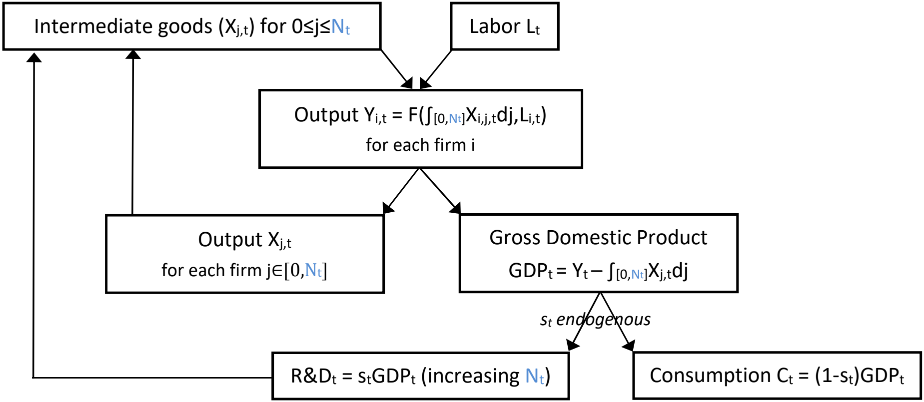
- **Final-good producers** use intermediate goods and labor to produce final goods.

- **Intermediate-good inventors-producers**
  - borrow from households to invent new types of intermediate goods,
  - use final goods to produce intermediate goods,
  - use their profits to reimburse households.

- **Households**
  - supply labor,
  - use final goods to consume and lend to inventors-producers.

- The **saving rate** (quantity of final goods lent / quantity of final goods consumed or lent) is **endogenous**, optimally chosen by households.

General overview of the model II



(In blue: stock; in black: flow.)

Final goods and intermediate goods

- **A single type** of final goods, used for
  - consumption,
  - the production of intermediate goods,
  - the invention of types of intermediate goods.

- **A continuum of types** of intermediate goods, used for the production of final goods.

- The assumption that
  - final goods are used to produce intermediate goods,
  - intermediate goods are used to produce final goods,

  is made for the sake of simplicity, not for the sake of realism.

- Replacing this assumption with a more realistic assumption would
  - substantially complicate the presentation of the model,
  - not qualitatively affect the results.

Private agents, markets

- **Three types** of private agents:
  - households,
  - final-good producers,
  - intermediate-good inventors-producers.

- We obtain that, for each type of intermediate goods, there is **only one inventor-producer of intermediate goods of this type** (hence imperfect competition).

- **Perfectly competitive markets**:
  - loan market,
  - labor market,
  - final-goods market.

- **Monopolistically competitive markets**: for each type of intermediate goods, the market of intermediate goods of this type.

Origin of supply and demand on each market

- **Loan market**:
  - *supply* from households,
  - *demand* from intermediate-good inventors-producers (to invent).

- **Labor market**:
  - *supply* from households,
  - *demand* from final-good producers.

- **Final-goods market**:
  - *supply* from final-good producers,
  - *demand* from households (pour consume and to lend) and intermediate-good inventors-producers (to produce).

- **Market of intermediate goods of type j**:
  - *supply* from the inventor-producer of intermediate goods of type j,
  - *demand* from final-good producers.

Exogenous variables

- **Neither flows nor stocks**:
  - continuous time, indexed by t,
  - price of final goods $=$ numéraire $\equiv 1$,
  - (large) number of final-good producers I.
- **Flow**:
  - labor supply $= 1$ per person.
- **Stocks**:
  - population $L > 0$ (constant over time),
  - initial size $N_0 > 0$ of the continuum of intermediate-good types.

Endogenous variables I

- **Prices** at time t:
  - real wage w_t,
  - real interest rate r_t,
  - real price $P_{j,t}$ of intermediate goods of type j.

- **Quantities – flows related to final-good producer** i at time t:
  - supply of final goods $Y_{i,t}$,
  - labor demand $L_{i,t}$,
  - demand $X_{i,j,t}$ of intermediate goods of type j.

- **Quantities – flows related to inventor-producer of intermediate goods of type** j at time t:
  - supply $X_{j,t}$ of intermediate goods of type j,
  - demand of final goods $Y_{j,t}$ (not to be confused with $Y_{i,t}$).

Endogenous variables II

- **Quantities** – **aggregate flows** at time t:
  - aggregate output of final goods $Y_t \equiv \sum_{i=1}^{I} Y_{i,t}$,
  - aggregate labor demand $L_t \equiv \sum_{i=1}^{I} L_{i,t}$,
  - aggregate consumption C_t.
- **Quantities** – **stocks** at time t:
  - size N_t of the continuum de intermediate-good types (except at $t = 0$),
  - real aggregate amount of loans B_t.

Chapter outline

Equilibrium conditions

Households' behavior

- Households are modeled exactly as in Chapter 2, with
  - a constant elasticity of intertemporal substitution, equal to $\frac{1}{\theta}$,
  - a population growth rate equal to zero ($n = 0$).

- Their behavior is thus characterized by the equilibrium conditions
  - $\dot{b}_t = w_t + r_t b_t - c_t$ (**instantaneous budget constraint**),
  - $\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\theta}$ (**Euler equation**),
  - $\lim_{t \to +\infty} \left\{ b_t e^{-\int_0^t r_\tau d\tau} \right\} = 0$ (**transversality condition**),

  where
  - $c_t \equiv \frac{C_t}{L}$ is per-capita consumption,
  - ρ is the rate of time preference ($\rho > 0$),
  - $b_t \equiv \frac{B_t}{L}$ is the aggregate amount of assets in units of goods per person.

Final-good-production function

- Production function for final-good producer i:

$$Y_{i,t} = F[L_{i,t}, (X_{i,j,t})_{0 \leq j \leq N_t}] \equiv A L_{i,t}^{1-\alpha} \int_0^{N_t} X_{i,j,t}^{\alpha} dj$$

where $A > 0$ and $0 < \alpha < 1$.

- So, for a given N_t, F has the same properties as in Chapters 1, 2 and 4.

- The "additively separable" form of F implies that the marginal product of $X_{i,j,t}$ is independent of $X_{i,j',t}$ for $j' \neq j$.

- So, **a new type of good is neither a direct substitute, nor a direct complement to existing types of good**.

Returns of intermediate goods

- $\frac{\partial^2 Y_{i,t}}{\partial X_{i,j,t}^2} < 0$: **the returns of each intermediate-good type are strictly decreasing**.

- If $\forall (j,j') \in [0;N_t]^2$, $X_{i,j,t} = X_{i,j',t} \equiv X_{i,t}$ (which will be the case in equilibrium), then $Y_{i,t} = AL_{i,t}^{1-\alpha} N_t X_{i,t}^{\alpha}$, so $\frac{\partial^2 Y_{i,t}}{\partial N_t^2} = 0$: **the returns of intermediate-good variety are constant**.

- The returns of intermediate goods are therefore
  - strictly decreasing in the intensive margin,
  - constant in the extensive margin.

- Technological progress, defined as the increase in N_t, will generate long-term growth thanks to the constant returns of N_t.

- Once invented, an intermediate-good type is never forgotten nor obsolete: as in Chapter 4, knowledge is cumulative.

Optimization problem of final-good producers

- There is no capital, and intermediate goods are non-durable, so the problem of final-good producers is **instantaneous**.
- The goods market is perfectly competitive, so each final-good producer takes the price of inputs as **given**.
- As a consequence, at each time t, the final-good producer i chooses $L_{i,t}$ and the $X_{i,j,t}$s so as to maximize their **instantaneous** profit

$$AL_{i,t}^{1-\alpha} \int_0^{N_t} X_{i,j,t}^{\alpha} dj - w_t L_{i,t} - \int_0^{N_t} P_{j,t} X_{i,j,t} dj$$

taking w_t, the $P_{j,t}$s and N_t as **given**.

First-order conditions

- The first-order conditions of this optimization problem are

$$w_t = (1-\alpha)\frac{Y_{i,t}}{L_{i,t}} \quad \text{(wage = marginal productivity of labor)},$$

$$X_{i,j,t} = \left(\frac{\alpha A}{P_{j,t}}\right)^{\frac{1}{1-\alpha}} L_{i,t} \quad \text{for any } j \in [0; N_t] \quad \text{(demand curves)}.$$

- Using these conditions to replace w_t and $P_{j,t}$ in the instantaneous profit, we get that this profit is zero for any $L_{i,t}$ and $X_{i,j,t}$.

Cost of an invention

- We assume that inventing dj new intermediate-good types (from N_t to $N_t + dj$) is a **deterministic process** requiring the use of ηdj units of final goods, where $\eta > 0$.

- This R&D cost is independent of N_t: we thus assume in particular that there is no depletion of new ideas.

- This cost is borrowed from households and reimbursed later at the real interest rate r_t.

Benefit of an invention I

- If the market for intermediate goods of a certain type were perfectly competitive, then no one would want to invent this type because the invention would be costly and would bring no benefit.
- We therefore assume that the inventor of a new intermediate-good type is rewarded by a **monopoly situation** on the market for intermediate goods of this type (due to a trade secret or a patent).
- Inventions are thus non-rival but excludable.
- For the sake of simplicity, we assume that an inventor-producer keeps their monopoly situation **forever**.
- Part 5 of the tutorials considers the alternative assumption of a **temporary** monopoly situation and studies the positive and normative implications.

Benefit of an invention II

- Production function for the inventor-producer of intermediate goods of type j (once this type invented): $X_{j,t} = Y_{j,t}$.

- The actualized benefit of inventing type j, at any time t posterior or equal to the invention time, is therefore

$$V_{j,t} = \int_t^{+\infty} (P_{j,\nu} - 1) X_{j,\nu} e^{-\int_t^{\nu} r_\tau d\tau} d\nu.$$

- As the market for type-j intermediate goods and the labor market clear,

$$\begin{aligned} X_{j,\nu} &= \sum_{i=1}^{I} X_{i,j,\nu} = \sum_{i=1}^{I} \left(\frac{\alpha A}{P_{j,\nu}}\right)^{\frac{1}{1-\alpha}} L_{i,\nu} \\ &= \left(\frac{\alpha A}{P_{j,\nu}}\right)^{\frac{1}{1-\alpha}} L_{\nu} = \left(\frac{\alpha A}{P_{j,\nu}}\right)^{\frac{1}{1-\alpha}} L. \end{aligned}$$

Benefit of an invention III

- Replacing $X_{j,\nu}$ with $\left(\frac{\alpha A}{P_{j,\nu}}\right)^{\frac{1}{1-\alpha}} L$ in the expression for $V_{j,t}$, we get

$$V_{j,t} = \int_t^{+\infty} (P_{j,\nu} - 1) \left(\frac{\alpha A}{P_{j,\nu}}\right)^{\frac{1}{1-\alpha}} L e^{-\int_t^\nu r_\tau d\tau} d\nu.$$

- At the type-j-invention time, denoted by t_j, the type-j inventor-producer chooses $(P_{j,\nu})_{\nu \geq t_j}$ taking into account the demand function $X_{j,\nu} = (\alpha A)^{\frac{1}{1-\alpha}} (P_{j,\nu})^{\frac{-1}{1-\alpha}} L$, as they are in a perpetual-monopoly situation on the market for type-j intermediate goods.

- They thus choose $(P_{j,\nu})_{\nu \geq t_j}$ at time t_j so as to maximize

$$V_{t_j} \equiv V_{j,t_j} = \int_{t_j}^{+\infty} (P_{j,\nu} - 1) \left(\frac{\alpha A}{P_{j,\nu}}\right)^{\frac{1}{1-\alpha}} L e^{-\int_{t_j}^\nu r_\tau d\tau} d\nu$$

taking $(r_\tau)_{\tau \geq t_j}$ as given.

Benefit of an invention IV

- This intertemporal problem amounts to an instantaneous problem: at each time $\nu \geq t_j$, the inventor-producer chooses $P_{j,\nu}$ so as to maximize

$$(P_{j,\nu} - 1)\left(\frac{\alpha A}{P_{j,\nu}}\right)^{\frac{1}{1-\alpha}} L.$$

- The first-order condition gives $P_{j,\nu} = \frac{1}{\alpha}$. **This price is**
    - **higher than the marginal cost** (because of the monopoly situation),
    - **constant over time** (like the instantaneous-profit function),
    - **constant across intermediate-good types** (by symmetry).

- In the limit case $\alpha \rightarrow 1$, which corresponds to perfect competition (intermediate goods then being perfectly substitutable with each other in the production function F), we have $P_{j,\nu} \rightarrow 1$ (price $\rightarrow$ marginal cost).

Benefit of an invention V

- Replacing $P_{j,t}$ with its value in $X_{j,t} = \left(\frac{\alpha A}{P_{j,t}}\right)^{\frac{1}{1-\alpha}} L$, we get

$$X_{j,t} = A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} L.$$

- **The quantity of intermediate goods is therefore the same for all types and constant over time.**

- Replacing $P_{j,\nu}$ and $X_{j,\nu}$ with their values in the expression of $V_{j,t}$, for $t \geq t_j$, we get

$$V_{j,t} = A^{\frac{1}{1-\alpha}} (1-\alpha) \alpha^{\frac{1+\alpha}{1-\alpha}} L \int_t^{+\infty} e^{-\int_t^\nu r_\tau d\tau} d\nu.$$

Preliminaries to the cost-benefit analysis

- Using $X_{i,j,t} = \left(\frac{\alpha A}{P_{j,t}}\right)^{\frac{1}{1-\alpha}} L_{i,t}$, $P_{j,t} = \frac{1}{\alpha}$ and $L_t = L$, we get

$$\begin{aligned} Y_t &\equiv \sum_{i=1}^{I} Y_{i,t} = \sum_{i=1}^{I} A L_{i,t}^{1-\alpha} \int_0^{N_t} X_{i,j,t}^{\alpha} dj \\ &= \sum_{i=1}^{I} A L_{i,t}^{1-\alpha} \int_0^{N_t} \left(\frac{\alpha A}{P_{j,t}}\right)^{\frac{\alpha}{1-\alpha}} L_{i,t}^{\alpha} dj \\ &= \sum_{i=1}^{I} A L_{i,t} \int_0^{N_t} \left(\alpha^2 A\right)^{\frac{\alpha}{1-\alpha}} dj \\ &= A^{\frac{1}{1-\alpha}} \alpha^{\frac{2\alpha}{1-\alpha}} L_t N_t = A^{\frac{1}{1-\alpha}} \alpha^{\frac{2\alpha}{1-\alpha}} L N_t. \end{aligned}$$

- We focus on equilibria in which V_t is constant over time; let V denote its value.

- We show in the appendix that, in this case, r_t is constant over time, equal to

$$r(V) \equiv A^{\frac{1}{1-\alpha}} (1-\alpha) \alpha^{\frac{1+\alpha}{1-\alpha}} \frac{L}{V}.$$

Cost-benefit analysis I

- We first consider the following two cases:

  1. Case $V > \eta$: then the loan market is not cleared (because of free entry, an infinite measure of inventors-producers want to borrow).
     $\hookrightarrow$ This case is therefore impossible.

  2. Case $V < \eta$: then
     - N_t is constant for $t \geq 0$ (as there is no invention ever),
     - so $Y_t = A^{\frac{1}{1-\alpha}} \alpha^{\frac{2\alpha}{1-\alpha}} L N_t$ is constant for $t \geq 0$,
     - so c_t is bounded for $t \geq 0$ (since $L c_t \leq Y_t$),
     - so $r(V) \leq \rho$ (because the Euler equation is $\frac{\dot{c}_t}{c_t} = \frac{r(V)-\rho}{\theta}$),
     - so $V \geq A^{\frac{1}{1-\alpha}} (1-\alpha) \alpha^{\frac{1+\alpha}{1-\alpha}} \frac{L}{\rho}$.

     $\hookrightarrow$ We rule out this case by restricting the analysis to parameter values such that $\rho \leq r \equiv (1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}} A^{\frac{1}{1-\alpha}} \frac{L}{\eta}$.

Cost-benefit analysis II

- So, we are necessarily in the remaining case:

  3. Case $V = \eta$: then $r_t = r$ for $t \geq 0$.

- So **the actualized benefit of an invention, at the invention time, is constantly equal to its cost**, and the real interest rate is constant over time.

- The inventor-producer of type-j intermediate goods therefore uses all their benefits until time $+\infty$ to reimburse their initial debt.

- So, at any time $t \geq t_j$, their debt is equal to the actualized value $V_{j,t}$ of their future benefits.

- Now, $V_{j,t}$ is constant over time, so their debt is also constant over time, equal to its initial value η.

Cost-benefit analysis III

- As the loan market clears, we therefore have

$$B_t = \int_0^{N_t} V_{j,t} dj = \eta N_t.$$

- As the benefits of inventors-producers are entirely transferred to households as debt reimbursement, the final-goods-market-clearing condition is

$$Y_t = Lc_t + N_t X_{j,t} + \eta \dot{N}_t.$$

Equilibrium determination

Endogen. variables (except N_t and c_t) as functions of N_t

- **Quantities:**
  - we have already obtained $L_t = L$ and $B_t = \eta N_t$,
  - we have $Y_t = A^{\frac{1}{1-\alpha}} \alpha^{\frac{2\alpha}{1-\alpha}} L N_t = \frac{\eta r N_t}{\alpha(1-\alpha)}$ and $X_{j,t} = A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} L = \frac{\alpha \eta r}{1-\alpha}$.
- **Prices:**
  - we have already obtained $r_t = r$ and $P_{j,t} = \frac{1}{\alpha}$,
  - from $w_t = (1-\alpha)\frac{Y_{i,t}}{L_{i,t}}$, we deduce that $\frac{Y_{i,t}}{L_{i,t}}$ is independent of i and is therefore equal to $\frac{Y_t}{L}$. As a consequence, $w_t = (1-\alpha)\frac{Y_t}{L} = \frac{\eta r N_t}{\alpha L}$.

Equilibrium conditions on N_t and c_t I

- Using $b_t \equiv \frac{B_t}{L} = \frac{\eta N_t}{L}$, $r_t = r$ and $w_t = \frac{\eta r N_t}{\alpha L}$, we can rewrite households' instantaneous budget constraint as

$$\dot{N}_t = \frac{(1+\alpha)r}{\alpha} N_t - \frac{L}{\eta} c_t.$$

- This differential equation can also be obtained by replacing Y_t with $\frac{\eta r N_t}{\alpha(1-\alpha)}$ and $X_{j,t}$ with $\frac{\alpha \eta r}{1-\alpha}$ in the final-goods-market-clearing condition (consequence of Walras' law).

Equilibrium conditions on N_t and c_t II

- Using $r_t = r$, we can rewrite the Euler equation as

$$\frac{\dot{c}_t}{c_t} = \frac{r - \rho}{\theta}.$$

- Using $b_t = \frac{\eta N_t}{L}$ and $r_t = r$, we can rewrite the transversality condition as

$$\lim_{t \to +\infty} \left\{ N_t e^{-rt} \right\} = 0.$$

Equilibrium conditions on N_t and c_t III

- $(N_t)_{t\geq 0}$ and $(c_t)_{t\geq 0}$ are thus determined by two differential equations, one initial condition and one terminal condition:

$$\dot{N}_t = \frac{(1+\alpha)r}{\alpha} N_t - \frac{L}{\eta} c_t,$$

$$\frac{\dot{c}_t}{c_t} = \frac{r-\rho}{\theta},$$

$$N_0 \text{ given},$$

$$\lim_{t \to +\infty} \left\{ N_t e^{-rt} \right\} = 0.$$

- The other endogenous variables are residually determined, from N_t, using their previously obtained expressions.

Determination of N_t and c_t I

- We integrate the differential equation in $\dot{c}_t$ to get

$$c_t = c_0 e^{\frac{r-\rho}{\theta} t}.$$

- The condition $r \geq \rho$ implies that the growth rate of per-capita consumption is non-negative.

- We restrict the analysis to parameter values such that $\rho > \frac{1-\theta}{\theta}(r-\rho)$, for intertemporal utility to take a finite value.

Determination of N_t and c_t II

- We can then rewrite the differential equation in $\dot{N}_t$ as

$$\dot{N}_t = \frac{(1+\alpha)r}{\alpha} N_t - \frac{Lc_0}{\eta} e^{\frac{r-\rho}{\theta} t}.$$

- Then, re-arranging the terms and multiplying by $e^{-\frac{(1+\alpha)r}{\alpha} t}$,

$$\left\{ \dot{N}_t - \frac{(1+\alpha)r}{\alpha} N_t \right\} e^{-\frac{(1+\alpha)r}{\alpha} t} = -\frac{Lc_0}{\eta} e^{-\varphi t},$$

where $\varphi \equiv \frac{(1+\alpha)r}{\alpha} - \frac{r-\rho}{\theta}$.

Determination of N_t and c_t III

- The condition $\rho > \frac{1-\theta}{\theta}(r-\rho)$ implies $r > \frac{r-\rho}{\theta}$ and hence

$$\varphi > \frac{r}{\alpha} > 0.$$

- We can therefore integrate the previous equality to get

$$N_t e^{-\frac{(1+\alpha)r}{\alpha}t} - N_0 = \frac{Lc_0}{\eta\varphi}e^{-\varphi t} - \frac{Lc_0}{\eta\varphi}$$

and then

$$N_t = \left(N_0 - \frac{Lc_0}{\eta\varphi}\right)e^{\frac{(1+\alpha)r}{\alpha}t} + \frac{Lc_0}{\eta\varphi}e^{-\left[\varphi - \frac{(1+\alpha)r}{\alpha}\right]t}.$$

Determination of N_t and c_t IV

- The transversality condition can the be rewritten as

$$\lim_{t \to +\infty} \left\{ \left(N_0 - \frac{Lc_0}{\eta\varphi} \right) e^{\frac{r}{\alpha}t} + \frac{Lc_0}{\eta\varphi} e^{-\left(\varphi - \frac{r}{\alpha}\right)t} \right\} = 0$$

and implies $c_0 = \frac{\eta\varphi N_0}{L} > 0$ since $\varphi > \frac{r}{\alpha} > 0$ (as in Chapters 2 and 4, c_0 adjusts to satisfy the transversality condition).

- We therefore finally obtain

$$N_t = N_0 e^{\frac{r-\rho}{\theta}t} \text{ and } c_t = \frac{\eta\varphi N_0}{L} e^{\frac{r-\rho}{\theta}t}.$$

Growth rate I

- So,
  - **product variety** N_t**,**
  - **per-capita consumption** c_t**,**
  - **per-capita final-good output** $y_t = \frac{\eta r N_t}{\alpha(1-\alpha)L}$**.**

  grow at the same constant rate.

- Defining GDP as the quantity of final goods produced minus the quantity of final goods used to produce intermediate goods:

$$GDP_t \equiv Y_t - \int_0^{N_t} Y_{j,t} dj = Y_t - N_t X_{j,t} = \frac{(1+\alpha)\eta r}{\alpha} N_t,$$

  we get that **GDP and per-capita GDP** $gdp_t \equiv \frac{GDP_t}{L}$ **also grow at this constant rate**.

Growth rate II

- **Because of the constant returns of N_t,**
  - **the growth rate is non-zero in the long term,**
  - **the convergence to steady state is instantaneous,**

  as in Romer's (1986) model, in which the social returns of capital are constant.

- This growth rate, equal to $\frac{r-\rho}{\theta}$ where $r \equiv A^{\frac{1}{1-\alpha}}(1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}\frac{L}{\eta}$, depends
  - positively on A, L and $\frac{1}{\theta}$,
  - negatively on η and ρ,
  - positively or negatively on α.

Growth rate III

- The sign of these derivatives can be interpreted in two steps:

1. $\frac{1}{\theta}\uparrow$ or $\rho\downarrow$ or $A\uparrow$ or $L\uparrow$ or $\eta\downarrow\Rightarrow$ loan supply $\uparrow$ for a given N_t:
   - $\frac{1}{\theta}\uparrow$ or $\rho\downarrow\Rightarrow$ loan supply $\uparrow$ for some given N_t and $r_t \Rightarrow$ loan supply $\uparrow$ for a given N_t because r_t remains such that actualized benefit = cost ($r_t = r$ is independent of $\frac{1}{\theta}$ and ρ);
   - $A\uparrow$ or $L\uparrow\Rightarrow$ marginal productivity $\uparrow\Rightarrow$ demand of intermediate goods of type $j\uparrow$ for a given $P_{j,t}\Rightarrow$ actualized benefit $\uparrow$ for a given $r_t \Rightarrow$ loan demand $\uparrow$ for some given N_t and $r_t \Rightarrow r_t\uparrow$ for a given N_t until actualized benefit = cost $\Rightarrow$ loan supply $\uparrow$ for a given N_t;
   - $\eta\downarrow\Rightarrow$ cost $\downarrow\Rightarrow$ loan demand $\uparrow$ for some given N_t and $r_t\Rightarrow r_t\uparrow$ for a given N_t until actualized benefit = cost $\Rightarrow$ loan supply $\uparrow$ for a given N_t;
2. loan supply $\uparrow$ for a given $N_t\Rightarrow$ loans $\uparrow$ for a given $N_t \Rightarrow \eta\dot{N}_t\uparrow$ for a given $N_t\Rightarrow\dot{N}_t\uparrow$ for a given N_t (as $\eta\rightarrow$ or $\downarrow$) $\Rightarrow$ growth rate $\frac{\dot{N}_t}{N_t}\uparrow$.

Stylised facts of Kaldor (1961)

- Unlike Romer's (1986) model, Romer's (1990) model accounts only for the 1st and 6th **stylised facts of Kaldor (1961)**, as capital is absent from this model:

  1. per-capita output grows: $\frac{\dot{y}_t}{y_t} = \frac{r-\rho}{\theta} \geq 0$,
  2. ~~the per-capita capital stock grows,~~
  3. ~~the rate of return of capital is constant,~~
  4. ~~the ratio capital / output is constant,~~
  5. ~~the labor and capital shares of income are constant,~~
  6. the growth rate of per-capita output varies across countries: $\frac{\dot{y}_t}{y_t} = \frac{r-\rho}{\theta}$ varies across countries when the preference parameters (ρ, θ) or scale parameters (L) vary across countries.

Scale effect

- The model predicts the existence of a **scale effect**: the larger the population size, the higher the growth rate.

- This scale effect seems not to be supported by empirical evidence at the country level. However, the model's population could correspond more to the world population than to the population of a country to the extent that
  - goods can circulate across countries,
  - a patent can be filed, or a trade secret be kept, in several countries.

- Kremer (1993) uncovers empirical evidence for a scale effect over long periods **at the world level**.

- **Michael Kremer**: American economist, born in 1964, professor at Harvard University since 1999, co-laureate (with Abhijit Banerjee and Esther Duflo) of the Sveriges Riksbank's prize in economic sciences in memory of Alfred Nobel in 2019 "*for their experimental approach to alleviating global poverty*".

Initial level and growth rate of c_t

- The initial level of per-capita consumption $c_0 = \frac{\eta \varphi N_0}{L}$ depends
  - positively on N_0, η and ρ,
  - negatively on L and $\frac{1}{\theta}$,
  - positively or negatively on A and α.

- c_0 and $\frac{\dot{c}_t}{c_t}$ react in opposite ways to a variation in L, η, ρ or $\frac{1}{\theta}$ in order to satisfy the intertemporal budget constraint.

Neither absolute convergence, nor conditional convergence

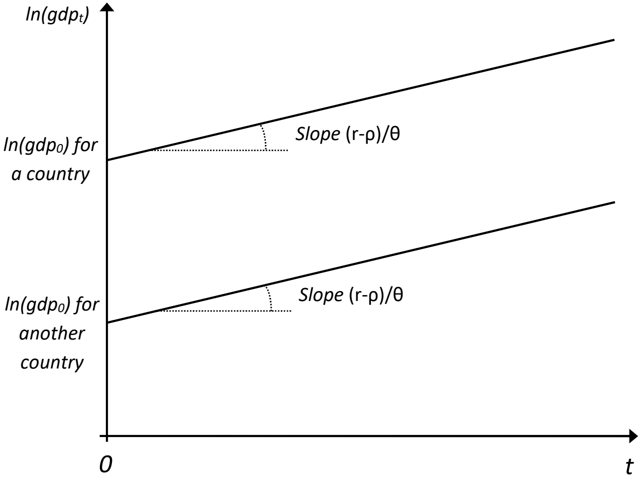
- We have $\ln(gdp_t) = \ln(gdp_0) + \frac{r-\rho}{\theta} t$, where $gdp_0 = \frac{(1+\alpha)\eta r}{\alpha L} N_0$.

- There is therefore no long-term convergence of $\ln(gdp_t)$ across countries that have different gdp_0s, even if they have the same
  - technological parameters A, α, η,
  - demographic parameter L,
  - preference parameters ρ, θ.

- The model therefore predicts **no absolute convergence and no conditional convergence** of $\ln(gdp_t)$ across countries, like Romer's (1986) model and unlike the Solow-Swan and Cass-Koopmans-Ramsey models.

- The fact that the absence of conditional convergence is not supported by empirical evidence (as seen in Chapter 1) is one more reason to consider this model as a model of the world economy.

No conditional convergence



Equilibrium sub-optimality

Social sub-optimality of the market equilibrium I

- The market equilibrium is socially optimal if and only if it coincides with the allocation chosen by the **benevolent, omniscient and omnipotent planner** $\mathcal{BOOP}$.

- Because of the strictly decreasing returns of each intermediate-good type, the $\mathcal{BOOP}$ orders the same quantity, denoted by X_t, for all intermediate-good types.

- The final-goods-market-clearing condition, which corresponds to the $\mathcal{BOOP}$'s resource constraint, is therefore

$$AL^{1-\alpha}X_t^{\alpha}N_t = Lc_t + X_tN_t + \eta\dot{N}_t.$$

Social sub-optimality of the market equilibrium II

- The $\mathcal{BOOP}$'s optimization problem is therefore the following: for a given $N_0 > 0$,

$$\max_{(c_t)_{t\geq 0},(X_t)_{t\geq 0},(N_t)_{t>0}} \int_0^{+\infty} e^{-\rho t} \left(\frac{c_t^{1-\theta} - 1}{1-\theta} \right) dt$$

subject to the constraints

1. $\forall t \geq 0$, $c_t \geq 0$ and $X_t \geq 0$ (non-negativity of consumptions),
2. $\forall t > 0$, $N_t \geq 0$ (non-negativity of the type-continuum size),
3. $\forall t \geq 0$, $\dot{N}_t \geq 0$ (type-continuum size not strictly decreasing over time)
4. $\forall t \geq 0$, $\dot{N}_t = \frac{(AL^{1-\alpha}X_t^{\alpha} - X_t)N_t - Lc_t}{\eta}$ (technology and resource constraint).

Social sub-optimality of the market equilibrium III

- We solve this problem as follows:
  1. we consider the problem obtained by ignoring the third constraint,
  2. we solve this auxiliary problem by applying the optimal-control theory, as in Chapter 2,
  3. we check that the solution to this auxiliary problem satisfies the third constraint of the initial problem.

- **Hamiltonian** associated to the auxiliary problem:

$$H^p(c_t, X_t, N_t, \lambda_t^p, t) \equiv e^{-\rho t}\left(\frac{c_t^{1-\theta}-1}{1-\theta}\right) + \lambda_t^p\left[\frac{(AL^{1-\alpha}X_t^{\alpha}-X_t)N_t - Lc_t}{\eta}\right]$$

where λ_t^p represents the value, measured in utility units at time 0, of an increase of η units of good in the resources at time t.

Social sub-optimality of the market equilibrium IV

- H^p is a function of
  - two control variables: c_t and X_t,
  - one state variable: N_t,
  - one costate variable: λ_t^p.

- We then get the following optimality conditions:
  - $\lambda_t^p = \frac{\eta}{L} e^{-\rho t} c_t^{-\theta}$ (first-order condition on c_t),
  - $X_t = \alpha^{\frac{1}{1-\alpha}} A^{\frac{1}{1-\alpha}} L$ (first-order condition on X_t),
  - $\dot{\lambda}_t^p = \frac{-(AL^{1-\alpha} X_t^{\alpha} - X_t)}{\eta} \lambda_t^p$ (costate equation),
  - $\dot{N}_t = \frac{(AL^{1-\alpha} X_t^{\alpha} - X_t) N_t - L c_t}{\eta}$ (resource constraint),
  - $\lim_{t \to +\infty} N_t \lambda_t^p = 0$ (transversality condition).

Social sub-optimality of the market equilibrium V

- Manipulating these conditions in the same way as in Chapters 2 and 4, we get
  - $\dot{N}_t = r^p N_t - \frac{L}{\eta} c_t$ (differential equation in $\dot{N}_t$),
  - $\frac{\dot{c}_t}{c_t} = \frac{r^p - \rho}{\theta}$ (differential equation in $\dot{c}_t$),
  - $\lim_{t \to +\infty} \left(N_t e^{-r^p t} \right) = 0$ (transversality condition),

  where $r^p \equiv (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} A^{\frac{1}{1-\alpha}} \frac{L}{\eta} = \alpha^{\frac{-1}{1-\alpha}} r > r$.

- These three conditions and N_0 determine $(N_t)_{t \geq 0}$ and $(c_t)_{t \geq 0}$.

(In red on this page: changes from page 36.)

Social sub-optimality of the market equilibrium VI

- We integrate the differential equation in $\dot{c}_t$ and get

$$c_t = c_0 e^{\frac{r^p - \rho}{\theta} t}.$$

- The result $r^p > r$ and the condition $r \geq \rho$ imply that the growth rate of per-capita consumption is non-negative.

- We restrict the analysis to parameter values such that $\rho > \frac{1-\theta}{\theta}(r^p - \rho)$, for intertemporal utility to take a finite value.

Social sub-optimality of the market equilibrium VII

- We can then rewrite the differential equation in $\dot{N}_t$ as

$$\dot{N}_t = r^p N_t - \frac{Lc_0}{\eta} e^{\frac{r^p - \rho}{\theta} t}.$$

- Then, rearranging the terms and multiplying by $e^{-r^p t}$,

$$\left(\dot{N}_t - r^p N_t \right) e^{-r^p t} = -\frac{Lc_0}{\eta} e^{-\varphi^p t},$$

where $\varphi^p \equiv r^p - \frac{r^p - \rho}{\theta}$.

Social sub-optimality of the market equilibrium VIII

- The condition $\rho > \frac{1-\theta}{\theta}\left(r^p - \rho\right)$ implies $r^p > \frac{r^p-\rho}{\theta}$ and hence
$$\varphi^p > 0.$$

- We can therefore integrate the previous equation to get
$$N_t e^{-r^p t} - N_0 = \frac{Lc_0}{\eta\varphi^p} e^{-\varphi^p t} - \frac{Lc_0}{\eta\varphi^p}$$
and then
$$N_t = \left(N_0 - \frac{Lc_0}{\eta\varphi^p}\right) e^{r^p t} + \frac{Lc_0}{\eta\varphi^p} e^{-(\varphi^p - r^p)t}.$$

Social sub-optimality of the market equilibrium IX

- We then rewrite the transversality condition as

$$\lim_{t\to+\infty}\left[\left(N_0-\frac{Lc_0}{\eta\varphi^p}\right)+\frac{Lc_0}{\eta\varphi}e^{-\varphi^p t}\right]=0,$$

which implies that $c_0=\frac{\eta\varphi^p N_0}{L}>0$ since $\varphi^p>0$ (as in Chapters 2 and 4, c_0 is chosen so as to satisfy the transversality condition).

- We therefore finally obtain

$$N_t=N_0 e^{\frac{r^p-\rho}{\theta}t} \quad \text{and} \quad c_t=\frac{\eta\varphi^p N_0}{L}e^{\frac{r^p-\rho}{\theta}t},$$

$$\text{and then} \quad y_t=\frac{\eta r^p N_0}{(1-\alpha)L}e^{\frac{r^p-\rho}{\theta}t} \quad \text{and} \quad gdp_t=\frac{\eta r^p N_0}{L}e^{\frac{r^p-\rho}{\theta}t}.$$

- We then check that $\forall t\geq 0$, $\dot{N}_t\geq 0$ (as $r^p>r\geq\rho$).

Social sub-optimality of the market equilibrium X

- These results differ from the previous ones, so **the market equilibrium is not socially optimal**.

- More precisely, **the market equilibrium is socially sub-optimal**: U_0 takes a value strictly lower in the market equilibrium than with the $\mathcal{BOOP}$.

- This last result, which can be easily checked with computations, comes from the fact that the $\mathcal{BOOP}$ does not choose the market-equilibrium allocation even though this allocation satisfies the constraints of their optimization problem.

- As we will see, **the social sub-optimality of the market equilibrium is due to monopolistic competition** (even though monopolistic competition is the source of technological-progress remuneration).

Social sub-optimality of the market equilibrium XI

- The growth rate of N_t, c_t, y_t and gdp_t is equal to $\frac{r^p-\rho}{\theta}$ with the $\mathcal{BOOP}$ and to $\frac{r-\rho}{\theta}$ in the market equilibrium.

- As $r^p > r$, **the growth rate is higher with the $\mathcal{BOOP}$ than in the market equilibrium**.

- This growth-rate difference is due to the presence of monopolistic competition in the decentralized environment.

- Monopolistic competition implies that the marginal product of X_t is higher than its marginal cost of production in the market equilibrium ($\frac{\partial GDP_t}{\partial X_t} = \frac{1-\alpha}{\alpha} N_t > 0$), while the two are equal to each other with the $\mathcal{BOOP}$ ($\frac{\partial GDP_t}{\partial X_t} = 0$).

Social sub-optimality of the market equilibrium XII

- So, the social returns $\frac{\partial GDP_t}{\partial N_t}$ of product variety are higher with the $\mathcal{BOOP}$ than in the market equilibrium: $\eta r^p > \frac{(1+\alpha)\eta r}{\alpha}$.
- So, the social returns $\frac{1}{\eta}\frac{\partial GDP_t}{\partial N_t}$ of R&D are higher with the $\mathcal{BOOP}$ than in the market equilibrium: $r^p > \frac{(1+\alpha)r}{\alpha}$.
- So, the social returns of R&D with the $\mathcal{BOOP}$ are higher than its private returns in the market equilibrium: $r^p > r$.
- So, investment $\eta \dot{N}_t$ in R&D, for a given N_t, is higher with the $\mathcal{BOOP}$ than in the market equilibrium: $\frac{\eta(r^p-\rho)}{\theta}N_t > \frac{\eta(r-\rho)}{\theta}N_t$.
- So, the growth rate $\frac{\dot{N}_t}{N_t}$ is higher with the $\mathcal{BOOP}$ than in the market equilibrium: $\frac{r^p-\rho}{\theta} > \frac{r-\rho}{\theta}$.

Social sub-optimality of the market equilibrium XIII

- With the $\mathcal{BOOP}$, compared to the market equilibrium,
  - the initial per-capita consumption level c_0 can be lower or higher: $\frac{\eta\varphi^p N_0}{L} \lesseqgtr \frac{\eta\varphi N_0}{L}$ depending on the values of α, η, θ, ρ, A and L,
  - it can be higher, even though initial investment in R&D $\eta\dot{N}_0$ is also higher, because $GDP_0 = Lc_0 + \eta\dot{N}_0$ is itself higher.

- By comparison, in Romer's (1986) model, with the $\mathcal{BOOP}$, compared to the market equilibrium,
  - it is necessarily lower because initial investment in capital $\dot{K}_0 + \delta K_0$ is higher while $GDP_0 = Lc_0 + \dot{K}_0 + \delta K_0$ is the same.

Optimal-equilibrium implementation

Implementation of the socially optimal equilibrium I

- The social sub-optimality of the market equilibrium gives a role to economic policy.

- In particular, as we will show, a fiscal authority can implement the $\mathcal{BOOP}$'s allocation in a decentralized way by
  - **subsidizing the purchase of intermediate goods** at a rate such that their effective price is equal to their production cost,
  - **financing this subsidy by a lump-sum tax** on households, which does not "distort" their choices.

- Let τ denote the subsidy rate: for each quantity $1 - \tau$ of intermediate goods purchased, a quantity τ is offered by the fiscal authority.

- The effective price of one unit of intermediate goods of type j, for final-good producers, is therefore $(1 - \tau)P_{j,t}$.

Implementation of the socially optimal equilibrium II

- The instantaneous profit of final-good producer i becomes

$$AL_{i,t}^{1-\alpha}\int_0^{N_t}X_{i,j,t}^{\alpha}dj - w_t L_{i,t} - \int_0^{N_t}(1-\tau)P_{j,t}X_{i,j,t}dj$$

and the second first-order condition of its maximization becomes

$$X_{i,j,t} = \left[\frac{\alpha A}{(1-\tau)P_{j,t}}\right]^{\frac{1}{1-\alpha}} L_{i,t}.$$

- The instantaneous profit of the inventor-producer of intermediate goods of type j becomes

$$(P_{j,\nu}-1)X_{j,\nu} = (P_{j,\nu}-1)\left[\frac{\alpha A}{(1-\tau)P_{j,\nu}}\right]^{\frac{1}{1-\alpha}} L$$

and the first-order condition of its maximization remains $P_{j,\nu} = \frac{1}{\alpha}$.

Implementation of the socially optimal equilibrium III

- With the same kind of computations and reasonings as previously, we then get in turn

(a) $X_{j,t} = (1-\tau)^{\frac{-1}{1-\alpha}} \frac{\alpha\eta r}{1-\alpha}$,

(b) $V_{j,t} = (1-\tau)^{\frac{-1}{1-\alpha}}$ ...
...$\eta r \int_t^{+\infty} e^{-\int_t^{\nu} r_\tau d\tau} d\nu$,

(c) $Y_t = (1-\tau)^{\frac{-\alpha}{1-\alpha}} \frac{\eta r}{\alpha(1-\alpha)} N_t$,

(d) $V_{j,t} = \eta$,

(e) $r_t = (1-\tau)^{\frac{-1}{1-\alpha}} r$,

(f) $B_t = \eta N_t$,

(g) $w_t = (1-\tau)^{\frac{-\alpha}{1-\alpha}} \frac{\eta r}{\alpha L} N_t$.

- Households' instantaneous budget constraint is

$$\dot{b}_t = w_t + r_t b_t - c_t - t_t$$

where t_t is the per-capita lump-sum tax.

Implementation of the socially optimal equilibrium IV

- Using the fiscal authority's budget constraint

$$Lt_t = \tau P_{j,t} X_{j,t} N_t = \tau \left(1-\tau\right)^{\frac{-1}{1-\alpha}} \frac{\eta r}{1-\alpha} N_t$$

and the results (e), (f), (g), we can rewrite households' instantaneous budget constraint as

$$\dot{N}_t = \frac{\left(1-\tau\right)^{\frac{-1}{1-\alpha}} \left(1-\tau-\alpha^2\right) r}{\alpha(1-\alpha)} N_t - \frac{L}{\eta} c_t.$$

- This differential equation can also be obtained using the results (a) and (c) to replace $X_{j,t}$ and Y_t in the final-goods-market-clearing condition (consequence of Walras' law).

Implementation of the socially optimal equilibrium V

- The Euler equation and the transversality condition are still $\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\theta}$ and $\lim_{t \to +\infty} \left(b_t e^{-\int_0^t r_\tau d\tau} \right) = 0$.

- Using (e) and (f), we can rewrite them as $\frac{\dot{c}_t}{c_t} = \frac{(1-\tau)^{\frac{-1}{1-\alpha}} r - \rho}{\theta}$ and $\lim_{t \to +\infty} \left[N_t e^{-(1-\tau)^{\frac{-1}{1-\alpha}} rt} \right] = 0$.

- $(N_t)_{t \geq 0}$ and $(c_t)_{t \geq 0}$ are therefore determined by the four conditions

$$\dot{N}_t = \frac{(1-\tau)^{\frac{-1}{1-\alpha}} \left(1 - \tau - \alpha^2\right) r}{\alpha(1-\alpha)} N_t - \frac{L}{\eta} c_t, \qquad N_0 \text{ given},$$

$$\frac{\dot{c}_t}{c_t} = \frac{(1-\tau)^{\frac{-1}{1-\alpha}} r - \rho}{\theta} \qquad \text{and} \qquad \lim_{t \to +\infty} \left[N_t e^{-(1-\tau)^{\frac{-1}{1-\alpha}} rt} \right] = 0.$$

Implementation of the socially optimal equilibrium VI

- These conditions are the same as those determining the paths $(N_t)_{t\geq 0}$ and $(c_t)_{t\geq 0}$ ordered by the $\mathcal{BOOP}$ if and only if $\tau = 1 - \alpha$.

- We easily check that, when $\tau = 1 - \alpha$, the other quantities (X_t, Y_t, y_t, GDP_t, gdp_t) take the same values as with the $\mathcal{BOOP}$.

- Therefore, this subsidy at rate $1 - \alpha$, financed by a lump-sum tax, implements the socially optimal equilibrium, and does so thanks to two effects:
  - it **corrects the monopolistic-competition inefficiency** by making the effective price of intermediate goods equal to their production cost: $(1 - \tau) P_{j,t} = 1$,
  - it **increases the remuneration of technological progress**, and hence the incentive to do some R&D, by increasing the inventors-producers' instantaneous profit (before debt reimbursement).

Implementation of the socially optimal equilibrium VII

- A fiscal authority *can also* implement the $\mathcal{BOOP}$'s allocation in a decentralized way by
  - **subsidizing the production of final goods** at a rate such that the demand for interm. goods is equal to the demand that would obtain if the price of intermediate goods were equal to their production cost,
  - **financing this subsidy by a lump-sum tax** on households, which does not "distort" their choices.

- A fiscal authority *cannot* implement the $\mathcal{BOOP}$'s allocation in a decentralized way by
  - **subsidizing R&D**,
  - **financing this subsidy by a lump-sum tax**,

  because such an economic policy has no effect on X_t: it tackles a *symptom* (the too small amount of R&D) but not the *cause* (the presence of monopolistic competition) of the social sub-optimality of the market equilibrium.

Conclusion

Main predictions of the model

- Long-term growth is positive thanks to the constant returns of product variety.
- Long-term growth depends on technological and preference parameters, as well as on the population size.
- There is neither absolute convergence, nor conditional convergence, of the per-capita-output levels (in logarithm) across countries.
- The market equilibrium is socially sub-optimal because of the presence of monopolistic competition.
- Economic policies, in the form of subsidies financed by lump-sum taxes, can implement the socially optimal equilibrium.

One limitation of the model

- Technological progress is modeled as the expansion of product variety, and not as the improvement in the quality of existing goods.

  $\hookrightarrow$ The Schumpeterian theory models growth as "creative destruction", i.e. as the invention of new goods that are direct substitutes to existing goods, which they make obsolete.

Appendix

Proof that $r_t = r(V)$

We show that $r_t = r(V)$ in four steps:

1. We write $V_t = A^{\frac{1}{1-\alpha}}(1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}} L \mathcal{A}_t$, where $\mathcal{A}_t \equiv \int_t^{+\infty} e^{-\int_t^\nu r_\tau d\tau} d\nu$.

2. Writing $\mathcal{A}_t = g(t,t)$ with $g(u,v) \equiv \int_u^{+\infty} e^{-\int_v^\nu r_\tau d\tau} d\nu$, we get
$\dot{\mathcal{A}}_t = \frac{\partial g}{\partial u}(t,t) + \frac{\partial g}{\partial v}(t,t) = -e^{-\int_t^t r_\tau d\tau} + r_t \int_t^{+\infty} e^{-\int_t^\nu r_\tau d\tau} d\nu = r_t \mathcal{A}_t - 1.$

3. Using $V_t = V$ for $t \geq 0$, we deduce from the first step that
$\mathcal{A}_t = \frac{V}{A^{\frac{1}{1-\alpha}}(1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}} L}$ for $t \geq 0$, and hence $\dot{\mathcal{A}}_t = 0$.

4. We deduce from the second and third steps that $r_t \mathcal{A}_t - 1 = 0$ and hence
$r_t = \frac{1}{\mathcal{A}_t} = A^{\frac{1}{1-\alpha}}(1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}} \frac{L}{V} = r(V)$ for $t \geq 0$.