A Model of Post-2008 Monetary Policy

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Abstract

We introduce banks and bank reserves into the basic New Keynesian model and allow the central bank to set both the interest rate on reserves (IOR rate) and the nominal stock of reserves. Our model can account, in qualitative terms, for three key features of US inflation during the recent zero-lower-bound (ZLB) episodes: no significant deflation, little inflation volatility, and no significant inflation following quantitative-easing policies. Crucial to this result is our assumption that demand for bank reserves got close to satiation, but did not reach full satiation. We introduce liquid government bonds into the model to reconcile our non-satiation assumption with the fact that Treasury-bill rates were often below the IOR rate during the ZLB episodes. Looking ahead, we explore the implications of our model for the normalization of monetary policy and its operational framework (floor system).

1 Introduction

Since 2008, the US economy has gone through two zero-lower-bound (ZLB) episodes, one during and after the Great Recession (from December 2008 to December 2015), the other during the COVID crisis (from March 2020 to March 2022). Broad facts about inflation and money-market rates during these ZLB episodes pose challenges to standard – New Keynesian (NK) or monetarist – models of monetary policy. On the one hand, as emphasized by Cochrane (2018), US inflation data during the 2008-2015 ZLB episode reflect neither the strong deflationary pressures nor the excessive inflation volatility that NK models would predict. The same observation can be made about the 2020-2022 ZLB episode.

On the other hand, monetarist doctrine, Cochrane (2018) suggests, faces the challenge of explaining why some large expansions of the Fed's balance sheet had no apparent inflationary effects during these episodes. Another challenge for monetarist models is that money-market rates were often below the interest rate on reserves (henceforth, the IOR rate) during these

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episodes. Many observers [e.g., Cochrane (2014)] take this fact as prima facie evidence that the Fed satiated the demand for reserves and, thus, as an argument against any model that tries to confront the data by invoking a monetary friction.

In this paper, we present a simple monetary-policy model that can qualitatively account for these key observations about US inflation and money-market rates during ZLB episodes, and we explore the model's implications for the normalization and operational framework of monetary policy. Our model adds a monetarist element to the standard NK model by assuming that bank reserves have a convenience yield − holding reserves can reduce banking costs in our model. The central bank in our model sets both the IOR rate and the nominal stock of bank reserves; these are two instruments that the Fed controls directly and has emphasized in its communication since 2008.[1]

In an earlier paper (Diba and Loisel, 2021), we showed that adding a (possibly small) monetary friction to the NK model leads to a resolution of some NK puzzles and paradoxes.[2] This resolution also offers a response to Cochrane's (2018) criticisms of the NK model about its counterfactual implications for inflation at the ZLB. As Gabaix (2020) points out, and we explain briefly in the text, these counterfactual implications ultimately arise because the NK model does not deliver local-equilibrium determinacy under a permanent interest-rate peg. Alternative models that deliver determinacy do not share the NK model's implausible implications about deflationary pressures and inflation volatility.[3] Our model delivers determinacy under a peg because our central bank sets the money supply, either exogenously or following a quantitative-easing (QE) rule that reacts to output and inflation.[4] In particular, setting the IOR rate determines the demand for real reserves in the steady-state equilibrium of our model, and this demand pins down the steady-state price level, given the outstanding nominal stock of reserves.

Having the central bank of our model set the money supply also serves to rule out the possibility of deflationary equilibrium paths analyzed by Benhabib et al. (2001a, 2001b). These paths are associated, in standard monetary-policy models, with an unintended deflationary steady state and a permanently binding ZLB constraint. By contrast, since the central bank sets the money supply in our model, we have a unique steady state, and inflation is equal to the money growth rate at this steady state. So, deflationary steady-state equilibria cannot arise in our setup if the central bank does not shrink the money supply.

Thus, ZLB episodes can be uneventful according to our model: they don't have to generate

[1]Bank reserves constitute the only central-bank liability in the cashless model that we present in the main text. We add household cash to our model in the Appendix.

[2]More specifically, we addressed the forward-guidance puzzle, the fiscal-multiplier puzzle, the reversal puzzle, the paradox of toil, and the paradox of flexibility.

[3]The connection is also implied by the analysis of Michaillat and Saez (2021). They develop a model (with relative wealth in the utility function) that delivers determinacy under a permanent peg, and they show that this model (among other properties) does not generate a severe deflation at the ZLB.

[4]Our determinacy result under an exogenous money supply only requires general assumptions like concavity and homogeneity of utility and production functions. Our determinacy result under QE rules (presented in the Appendix) is obtained for an iso-elastic production function and holds, we argue, for any reasonable calibration.

severe deflationary pressures or excessive inflation volatility. Cochrane's (2018) criticism of monetary-policy models, however, is two-pronged. He argues against NK models because they imply severe deflationary pressures and inflation volatility, but also against monetarist doctrine because we did not observe a significant inflationary response to the Fed's massive balance-sheet expansions – which, Cochrane (2018) observes, seems to cast doubt on models that emphasize the role of a stable money-demand equation. Another challenge to our monetarist perspective on ZLB episodes is the fact that the federal-funds rate and T-bill returns dropped below the IOR rate during these episodes. If this fact signals satiation of money demand, we lose the link (between the money supply and the price level) that is central to monetarist doctrine, as Cochrane (2014) and others have noted.

We show that our model, with its monetarist element, can meet both challenges: we can generate small inflationary responses to large monetary expansions as well as money-market rates below the IOR rate. Our model can address these points because it gives an explicit role to bank reserves.[5] More specifically, we assume that holding reserves reduces, for banks, the costs associated with making loans. And to generate a demand for bank loans, we assume that firms need to pay in advance their wage bill (or some fraction of it).

Under two key conditions, our model implies a weak connection between the money supply and the price level, and can thus account for the absence of significant inflation following QE policies. These two conditions are: (1) the demand for reserves is "close to satiation" in a sense that we will articulate, and (2) the monetary expansion is perceived as temporary. More specifically, we conduct non-linear simulations of QE policies in our model, under a calibration to US data in November 2010 (i.e., at the start of the Fed's second round of QE). We find that large increases in the money supply (say, doubling the stock of reserves) can have very small inflationary effects (around twenty basis points per annum) if balance-sheet normalization is expected to occur in about five years and the marginal convenience yield of reserves is ten basis points per annum. Our simulations suggest strongly decreasing returns to QE: much larger monetary expansions also have fairly small inflationary effects in our (non-linear) numerical simulations, as long as policy normalization is expected to occur in, say, five years.

Our model can generate money-market rates below the IOR rate if non-bank entities have a strong demand for money-market assets (e.g., holding T-bills as collateral, or in response to regulatory constraints). To make this point, we propose an extension of our benchmark model that allows T-bills to have a lower return than reserves without requiring that demand for reserves be fully satiated. In our extended model, workers get utility from holding government bonds, as a proxy for pension funds and money-market funds that hold bonds (in reality) and provide financial services to households. The banks in our extended model can use bonds instead of reserves for liquidity management, but they choose not to do so in equilibrium; so,

[5] Standard models – e.g., with money in the utility function – do not have enough detail and structure to capture our points about demand for reserves and T-bills.

the equilibrium of this extended model coincides with the equilibrium of our benchmark model, except for T-bill returns. Adding government bonds with liquidity services, thus, enables our model to account for T-bill returns below the IOR rate without altering any of the implications for inflation.

US inflation behaved in a similar way during the 2008-2015 ZLB episode and the first half of the 2020-2022 ZLB episode. During the second half of the latter episode (and beyond), however, the US economy experienced a surge in inflation. Adverse supply shocks were presumably a major factor leading to this surge in inflation. Our model, however, suggests that monetary policy may have played a role as well. In particular, if large and repeated balance-sheet expansions eventually came to be viewed as an indication of persistently large reserve balances for the foreseeable future, our model would predict a rise in inflation in response to expansions that were previously viewed as temporary. In contrast to a model with full satiation of money demand, our model links the price level to the supply and demand for money in the long run. Persistent balance-sheet expansions may need to be accompanied by a rise in the IOR rate or other policies that increase money demand in our model.

Looking ahead, we explore the implications of our model for the normalization and operational framework of monetary policy. We find unambiguously negative effects of monetary-policy normalization on inflation: current and expected future IOR-rate hikes and balance-sheet contractions are always deflationary. Our result is in contrast to the Neo-Fisherian implication of some equilibria in NK models – discussed in Schmitt-Grohé and Uribe (2017) and Bilbiie (2022), among others – that suggest policy-rate hikes may raise inflation to target in economies that suffer from deflationary pressures. In our setup, the policy rate is the IOR rate, and expected policy-rate hikes exert deflationary pressures by increasing the demand for reserves, given the path for the nominal stock of reserves.

We consider a "floor system" in which the central bank sets the nominal stock of reserves exogenously and follows a rule for setting the IOR rate. Such a system, arguably, captures the Fed's intentions for the future operational framework of monetary policy (see, e.g., Federal Open Market Committee, 2019). We show that, in our model, this system delivers local-equilibrium determinacy for any non-negative response of the IOR rate to current inflation, and for a wide range of non-negative responses of the IOR rate to current output. The Taylor principle, thus, is not needed to ensure determinacy under a floor system.

All the results we present in this paper presume that bank reserves have a convenience yield. In reality, the convenience yield may arise from the banking sector's need for liquidity management or preference for safe assets; it may also reflect the usefulness of reserves for compliance with regulatory constraints (like liquidity-coverage requirements) and bank strategies for passing stress tests.[6] Whatever the source of the convenience yield may be in reality, its presence is central to

[6] Afonso et al. (2020) discuss the increase in demand for reserves reflecting Basel III regulations.

our analysis. Our arguments would fall apart if we assumed full satiation of demand for reserves.

Empirical evidence presented by Ennis and Wolman (2015) and Reis (2016) does not support the satiation view during QE1. Reis's (2016) evidence, however, does not reject the satiation hypothesis during QE2 and QE3, when large increases in reserve balances had no apparent effect on expected inflation. This type of evidence is also the gist of Cochrane's (2018) criticism of monetarist doctrine, as we noted above. Our counter-argument, based on our numerical simulation of QE policies, is that this evidence may also be consistent with demand for reserves being close to satiation, rather than fully satiated. The distinction between these two possibilities (close-to-satiation versus fully satiated demand) does matter for monetary models; indeed, the implications of these models change discontinuously as we go from arbitrarily small convenience yields to a literal interpretation of full satiation. The distinction, however, may be difficult to make in practice. For example, in contrast to Reis's (2016) evidence about expected inflation, Krishnamurthy and Lustig (2019) find statistically significant effects of monetary policy, during and after QE2, on the convenience yield of US Treasury bills and the foreign-exchange value of the dollar. Copeland et al. (2022) present evidence of an apparent excess demand for reserves, despite a large Fed balance sheet, during the "balance-sheet normalization" phase from December 2017 to September 2019.

Our work is complementary to analyses of QE – e.g., Gertler and Karadi (2011), Sims et al. (2023) – that highlight the asset side of the central bank's balance sheet, to focus on credit-market frictions. We highlight the liability side, to focus on inflation and money-market rates. Our modeling of reserves and banking costs is similar to the ones in Cúrdia and Woodford (2011) and Ireland (2014); however, our modeling choices enable us to obtain new analytical results.[7] There is, moreover, no overlap between the topics of these two papers and our focus on reconciling the model with "broad facts" about ZLB episodes.

Our focus on these broad facts is also what distinguishes our work from more recent contributions that add a banking sector to the NK framework. Ennis (2018) develops a model in which the central bank sets the IOR rate; he highlights alternative cases in which the price level "decouples" from the monetary base (with market rates equal to the IOR rate), or moves one-for-one with the base (with market rates below the IOR rate). Arce et al. (2020) and Bigio and Sannikov (2021) develop models with frictions in the interbank market. These two papers provide better microfoundations for the liquidity services of reserves. Ulate (2021) and Eggertsson et al. (2023) analyze the effects of negative policy rates in models with banks. Piazzesi et al. (2022) highlight the transmission of changes in the IOR rate to other interest rates, and compare corridor and

[7] Our new analytical results are the proof of local-equilibrium determinacy and the derivation of the solution for inflation and output under exogenous policy instruments. Our modeling of a production function for banks and a cost channel of monetary policy is closer to Ireland (2014). Compared to Cúrdia and Woodford (2011), the differences in modeling choices are the following: (i) we link banking costs to time spent on banking activities; (ii) the borrowers in our model are firms (borrowing the wage bill or some fraction of it); and (iii) we assume that demand for reserves is not satiated.

floor systems. Benigno and Benigno (2022) characterize optimal changes in reserves and the IOR rate in a liquidity trap and during the policy normalization process.

In terms of analytical results, the main overlap between our paper and other NK models with banks is between our analysis of local determinacy and the analysis in Piazzesi et al. (2022).[8] In a broader sense, however, the central feature that makes the transmission mechanism of monetary policy, in all of these models, different from the standard NK mechanism is the idea that the policy rate reflects a "convenience yield," or a liquidity premium that policy can affect.[9] Earlier contributions that highlight the theoretical relevance of such a convenience yield, without explicitly modeling banks, include Andolfatto (2015), Benigno and Nisticò (2017), Canzoneri and Diba (2005), and Hagedorn (2018).[10]

Our approach to allowing for the convenience yields of bank reserves and government bonds boils down to putting them in the utility function of a representative household. A sizeable literature explores the microfoundations of this approach. Geromichalos and Herrenbrueck (2022) summarize and extend this literature. Williamson (2012) discusses the Great Recession from the perspective of this literature. Andolfatto and Williamson (2015) develop a model with segmented markets in which some agents can use bonds (as well as money) in exchange. This provides better microfoundations – than our bonds-in-the-utility-function approach – for imparting a liquidity premium on bonds.

The rest of the paper is organized as follows. Section 2 presents our benchmark model. Sections 3 and 4 show that this model can account for the three features, discussed above, of US inflation during ZLB episodes. Section 5 briefly discusses how an extended model with liquid government bonds can account not only for these three features of inflation, but also for the negative spread between T-bill and IOR rates observed during ZLB episodes (with the detailed analysis relegated to the Appendix). Section 6 explores the implications of our model for the normalization and operational framework of monetary policy, and Section 7 concludes.

2 Benchmark Model

In this section, we present our benchmark model. This model will be extended with household cash in Appendix D, and with liquid government bonds in Appendix E.

In our benchmark model, monopolistically competitive firms use labor to produce goods. They need to pay the wage bill (or some fraction of it) before they can produce and sell their output.

[8] The two papers have the same local determinacy result when reserves are exogenous, but different ways of allowing for some endogeneity.

[9] The endogenous convenience yield generates portfolio-balance effects when the money (or bond) supply changes.

[10] Andolfatto (2015) also addresses the behavior of US inflation after the Great Recession, linking low inflation to satiation of demand for reserves. Our goal in Section 4 is to generate low inflation without assuming full satiation of demand for reserves.

They borrow the corresponding amount from banks. Banks incur costs making loans, and holding reserves mitigates these costs. The central bank sets both the interest rate on bank reserves and the quantity of bank reserves. We merge households and banks in our model, as there are no frictions between them. We make standard assumptions on utility and production functions (like monotonicity and concavity), without specifying any functional form.

2.1 Households (Reduced-Form Setup)

Each household consists of production workers and bankers. In this first subsection, we start from households' *reduced-form* utility function, whose arguments are consumption (c_t), hours worked by production workers (h_t), real loans (ℓ_t), and real reserves (m_t):

$$U_t = \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} \beta^k \zeta_{t+k} \left[u\left(c_{t+k}\right) - v\left(h_{t+k}\right) - \Gamma\left(\ell_{t+k}, m_{t+k}\right) \right] \right\}, \tag{1}$$

where $\beta \in (0,1)$ and ζ_t denotes a stochastic exogenous discount-factor shock of mean one. The consumption-utility function u, defined over the set of positive real numbers $\mathbb{R}_{>0}$, is twice differentiable, strictly increasing ($u' > 0$), strictly concave ($u'' < 0$), and satisfies the usual Inada conditions $\lim_{c_t \to 0} u'(c_t) = +\infty$ and $\lim_{c_t \to +\infty} u'(c_t) = 0$. The labor-disutility function v, defined over the set of non-negative real numbers $\mathbb{R}_{\geq 0}$, is twice differentiable, strictly increasing ($v' > 0$), and weakly convex ($v'' \geq 0$).

The term $-\Gamma(\ell_t, m_t)$ in (1) comes from households acting as bankers. In Subsection 2.5, we will articulate how bankers produce loans using reserves and their own labor effort as inputs, and we will specify the *primitive* utility function of households. In the present subsection, we take a lighter approach to convey intuition: we simply work with the implied utility cost of making loans $\Gamma\left(\ell_t, m_t\right)$, reflecting bankers' disutility from work. This utility cost of banking is strictly increasing in loans ($\Gamma_\ell > 0$) because bankers have to work harder to make more loans. It is strictly decreasing in reserves ($\Gamma_m < 0$) because holding reserves reduces the labor effort needed to make a given amount of loans. It is also convex ($\Gamma_{\ell\ell} > 0$, $\Gamma_{mm} > 0$, $\Gamma_{\ell\ell}\Gamma_{mm} - (\Gamma_{\ell m})^2 \geq 0$), and such that $\Gamma_{\ell m} < 0$ (which says that a marginal increase in reserves decreases costs by more the larger are loans). Finally, it satisfies the limit properties $\lim_{m_t \to +\infty} \Gamma_m(\ell_t, m_t) = 0$ and $\lim_{m_t \to 0} \Gamma_\ell(\ell_t, m_t) = +\infty$ for any $\ell_t \in \mathbb{R}_{>0}$. The former property is a standard Inada condition, while the latter articulates a sense in which holding reserves is essential for banking.

In addition to making loans and holding reserve balances at the central bank, households trade bonds b_t (in zero net supply). Loans, reserves, and bonds are one-period non-contingent assets. We let I_t^ℓ, I_t^m, and I_t denote the corresponding gross nominal interest rates. We let P_t denote the price level, and $\Pi_t \equiv P_t / P_{t-1}$ the gross inflation rate. The household budget constraint, expressed in real terms, is then

$$c_t + b_t + \ell_t + m_t \leq \frac{I_{t-1}}{\Pi_t} b_{t-1} + \frac{I_{t-1}^\ell}{\Pi_t} \ell_{t-1} + \frac{I_{t-1}^m}{\Pi_t} m_{t-1} + w_t h_t + \tau_t, \tag{2}$$

where w_t represents the real wage and τ_t captures firm profits and lump-sum taxes or transfers.

Households choose b_t, c_t, h_t, ℓ_t, and m_t to maximize their reduced-form utility function (1) subject to their budget constraint (2), taking all prices (I_t, I_t^ℓ, I_t^m, P_t, and w_t) as given. Letting λ_t denote the Lagrange multiplier on the period-t budget constraint, the first-order conditions of the household optimization problem are

$$\lambda_t = \zeta_t u'(c_t), \tag{3}$$

$$\lambda_t = \beta I_t \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\}, \tag{4}$$

$$\lambda_t w_t = \zeta_t v'(h_t), \tag{5}$$

$$\zeta_t \Gamma_\ell (\ell_t, m_t) + \lambda_t = \beta I_t^\ell \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\},$$

$$\zeta_t \Gamma_m (\ell_t, m_t) + \lambda_t = \beta I_t^m \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\}.$$

Using (4), we can rewrite the last two conditions as

$$\frac{I_t^\ell}{I_t} = 1 + \frac{\zeta_t \Gamma_\ell (\ell_t, m_t)}{\lambda_t}, \tag{6}$$

$$\frac{I_t^m}{I_t} = 1 + \frac{\zeta_t \Gamma_m (\ell_t, m_t)}{\lambda_t}. \tag{7}$$

Condition (6) implies that loans pay more interest than bonds, because the marginal banking cost is positive ($\Gamma_\ell > 0$). Condition (7) implies that reserves pay less interest than bonds, because they serve to reduce banking costs ($\Gamma_m < 0$). The household optimization problem is also subject to a standard no-Ponzi-game constraint, and the transversality condition is

$$\lim_{k \to +\infty} \mathbb{E}_t \left\{ \beta^{t+k} \lambda_{t+k} a_{t+k} \right\} = 0, \tag{8}$$

where $a_t \equiv b_t + \ell_t + m_t$ denotes the household's total assets. The second-order conditions of the optimization problem are met because of the convexity of Γ.

2.2 Firms

There is a continuum of monopolistically competitive firms owned by households and indexed by $i \in [0,1]$. Firm i uses $h_t(i)$ units of labor to produce

$$y_t(i) = f[h_t(i)] \tag{9}$$

units of output. The production function f, defined over $\mathbb{R}_{\geq 0}$, is twice differentiable, strictly increasing ($f' > 0$), and weakly concave ($f'' \leq 0$); it also satisfies $f(0) = 0$. To generate a demand for bank loans, we assume that firm i has to borrow a fraction $\phi \in (0,1]$ of its nominal wage bill $W_t h_t(i)$ from banks, at the gross nominal interest rate I_t^ℓ, before it can produce and sell its output. Thus, the nominal value of firm i's loan $L_t(i)$ must satisfy

$$\phi W_t h_t(i) \leq L_t(i). \tag{10}$$

Following Calvo (1983), we assume that at any given date, each firm (whatever its pricing history may be) is not allowed to reset its price with probability $\theta \in [0,1)$. If allowed to reset its price at date t, firm i chooses its new price $P_t^*(i)$ to maximize the expected present value of the profits that this price will generate:

$$\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \frac{\lambda_{t+k}}{\lambda_t \Pi_{t,t+k}} \left[P_t^*(i)\, y_{t+k}(i) - \frac{\beta \lambda_{t+k+1} I_{t+k}^{\ell} L_{t+k}(i)}{\lambda_{t+k} \Pi_{t+k+1}} - \left[W_{t+k} h_{t+k}(i) - L_{t+k}(i) \right] \right] \right\}.$$

The two negative terms inside the (large) square brackets represent production costs at date $t+k$. There are two terms because firms borrow a fraction of the wage bill: the first term represents the value at date $t+k$ of loan repayments made at date $t+k+1$; the second term is the part of the wage bill at date $t+k$ that is not borrowed. The optimization is subject to the production function (9), the borrowing constraint (10), and the demand schedule

$$y_{t+k}(i) = \left[\frac{P_t^*(i)}{P_{t+k}} \right]^{-\varepsilon} y_{t+k}, \tag{11}$$

where $\Pi_{t,t+k} \equiv P_{t+k}/P_t$ for any $k \in \mathbb{N}$, $\varepsilon > 0$ denotes the elasticity of substitution between differentiated goods, and $y_t \equiv [\int_0^1 y_t(i)^{(\varepsilon-1)/(\varepsilon)} di]^{\varepsilon/(\varepsilon-1)}$.

Since the household first-order condition (6) implies $I_t^{\ell} > I_t$, the borrowing constraint of firms (10) is binding. Using the Euler equation (4) and the law of iterated expectations, we can write the first-order condition for the firm's optimization problem as

$$\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \frac{\lambda_{t+k}}{\lambda_t \Pi_{t,t+k}} \left[P_t^*(i) - \left(\frac{\varepsilon}{\varepsilon - 1} \right) \left(\phi \frac{I_{t+k}^{\ell}}{I_{t+k}} + (1-\phi) \right) \frac{W_{t+k}}{f'\left[h_{t+k}(i)\right]} \right] y_{t+k}(i) \right\} = 0. \tag{12}$$

In the particular case of flexible prices ($\theta = 0$), and in a symmetric equilibrium (with $P_t^*(i) = P_t$ and $h_t(i) = h_t$), this first-order condition becomes

$$P_t = \frac{\varepsilon}{\varepsilon - 1} \left[\phi \frac{I_t^{\ell}}{I_t} + (1-\phi) \right] \frac{W_t}{f'(h_t)}. \tag{13}$$

2.3 Government

For simplicity, our benchmark model abstracts from government expenditures and government bonds. Introducing government expenditures into the model would not affect our results in any substantive way as long as these expenditures do not enter households' utility function or enter it in a separable way (as is standard in the literature). Government bonds would not matter at all if they served only as a store of value, but may matter if they provide liquidity services; we will introduce liquid government bonds into the model in Section 5 and Appendix E.

The central bank has two independent instruments: the (gross) nominal interest rate on reserves I_t^m, and the stock of nominal reserves M_t. Absent government bonds, the central bank injects reserves via lump-sum transfers (T_t). The nominal stock of reserves thus evolves according to

$$M_t = I_{t-1}^m M_{t-1} + T_t. \tag{14}$$

To capture a lower bound on I_t^m in a simple and stark way, we assume that vault cash (with no interest payments) is a perfect substitute for deposits at the central bank in terms of reducing banking costs. This introduces a zero lower bound (ZLB) for the net nominal IOR rate $I_t^m - 1$ in our model.[11] In an equilibrium with $I_t^m > 1$, banks will hold no cash. In an equilibrium with $I_t^m = 1$, the composition of reserve balances will be indeterminate, but also inconsequential; so, we will assume that banks hold no cash in equilibrium.

2.4 Market Clearing

The bond-market, reserve-market and goods-market clearing conditions are respectively

$$b_t = 0, \tag{15}$$

$$m_t = \frac{M_t}{P_t}, \tag{16}$$

$$c_t = y_t. \tag{17}$$

2.5 Households (Primitive Setup)

In this subsection, we briefly describe the primitive setup for households that leads to the reduced-form utility function (1). This brief description will be useful when we calibrate the model later in the paper (in Subsection 4.1). In this primitive setup, households get utility from consumption (c_t) and disutility from labor (h_t for production workers, h_t^b for bankers). Their intertemporal utility function is

$$U_t = \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} \beta^k \zeta_{t+k} \left[u\left(c_{t+k}\right) - v\left(h_{t+k}\right) - v^b\left(h_{t+k}^b\right) \right] \right\}.$$

Like v, the labor-disutility function v^b is defined over $\mathbb{R}_{\geq 0}$, twice differentiable, strictly increasing ($v^{b\prime} > 0$), and weakly convex ($v^{b\prime\prime} \geq 0$). Bankers use their own labor h_t^b and (real) reserves at the central bank m_t to produce (real) loans ℓ_t according to the technology

$$\ell_t = f^b\left(h_t^b, m_t\right).$$

The production function f^b, defined over $\left(\mathbb{R}_{\geq 0}\right)^2$, is twice differentiable, strictly increasing ($f_h^b > 0$ and $f_m^b > 0$), homogeneous of degree $d \in (0, 1]$, and such that $f_{hh}^b < 0$, $f_{mm}^b < 0$, and $f_{hm}^b \geq 0$. These assumptions imply that f^b is concave, as we show in Appendix A.1. In addition, we assume that for any $h_t^b \in \mathbb{R}_{\geq 0}$, $\lim_{m_t \to +\infty} f_m^b(h_t^b, m_t) = 0$ and $\lim_{m_t \to 0} f_h^b(h_t^b, m_t) = 0$. The former assumption is a standard Inada condition, while the latter articulates a sense in which holding reserves is essential for banking.

[11] A more realistic model in which vault cash is substitutable to some extent for deposits at the central bank could imply a negative lower bound for the net nominal IOR rate. Whether the effective lower bound is zero or negative does not matter for most of our analysis below.

The function f^b is, of course, a convenient short cut to capture the role of bank reserves − which in reality may come, for example, from a maturity mismatch between banks' assets and liabilities. For the sake of generality, we do not impose any functional form for f^b. Examples of functional forms satisfying all the assumptions listed above include, in particular, constant-elasticity-of-substitution (CES) functions, and more generally (but not exclusively) CES functions raised to a power d such that $\max[(s-1)/s, 0] < d \leq 1$, where s denotes the elasticity of substitution. Moreover, we could relax some assumptions to some extent without affecting our results − for example, the assumption that labor and reserves are complements ($f^b_{hm} \geq 0$), or the assumption of decreasing or constant returns to scale ($d \leq 1$), which we make to simplify our analysis.[12]

Since $f^b_h > 0$, we can invert the production function of bankers f^b and get their labor hours as a function of loans and reserves: $h^b_t = g^b(\ell_t, m_t)$, where g^b is implicitly and uniquely defined by $\ell_t = f^b[g^b(\ell_t, m_t), m_t]$. Using this result to eliminate h^b_t in the primitive utility function, we get the reduced-form utility function (1), with the utility cost of banking

$$\Gamma\left(\ell_t, m_t\right) \equiv v^b\left[g^b\left(\ell_t, m_t\right)\right].$$

We establish some useful properties of the function g^b in Appendix A.2, and we show in Appendix A.3 that the function Γ has all the properties mentioned in Subsection 2.1.

3 Inflation at the ZLB

In this section, we show that our benchmark model has a unique steady state under constant monetary-policy instruments, and delivers local-equilibrium determinacy under exogenous monetary-policy instruments. We explain how this determinacy result can account for two features of US inflation during ZLB episodes: no significant deflation, and little inflation volatility. At the end of the section, we briefly discuss (relegating the detailed analysis to Appendices C and D) how this determinacy result is essentially robust to the relaxation of two simplifying assumptions in turn: the exogeneity of nominal reserves, and the absence of household cash.

3.1 Unique Steady State

For our steady-state analysis, we assume that the IOR rate I^m_t is permanently pegged ($I^m_t = I^m \geq 1$), the stock of nominal reserves is constant over time ($M_t = M > 0$), and there are no discount-factor shocks ($\zeta_t = 1$).[13] In any steady state, real reserves are constant over time (by definition of a steady state), and so are nominal reserves (by assumption); therefore, prices are also constant over time, and the flexible-price version of firms' first-order condition (13) holds.

[12] In an earlier version of the paper, we allowed for increasing returns to scale ($d > 1$) when the function f^b is iso-elastic.

[13] Our results would be unchanged if we assumed that the stock of nominal reserves grows or shrinks at a constant rate and that non-optimized prices are indexed to steady-state inflation.

We first use the equilibrium conditions (3), (5), (9), (10) holding with equality, and (17) to express steady-state loans ℓ as a function of steady-state employment h:

$$\ell = \mathcal{L}(h) \equiv \frac{\phi h v'(h)}{u'[f(h)]}. \tag{18}$$

The function $\mathcal{L}$, defined over $\mathbb{R}_{>0}$, is strictly increasing ($\mathcal{L}' > 0$), with $\lim_{h\to 0} \mathcal{L}(h) = 0$ and $\lim_{h\to +\infty} \mathcal{L}(h) = +\infty$. The reason is simply that loans are proportional to the wage bill (wage times employment), and the wage is increasing in employment.

Next, in Appendix A.4, we show that the equilibrium conditions (3), (5), (6), (9), (13), (17), and (18) implicitly and uniquely define a function $\mathcal{M}$ relating steady-state real reserves m to steady-state employment h:

$$m = \mathcal{M}(h). \tag{19}$$

This function is strictly increasing ($\mathcal{M}' > 0$). The reason is that in any steady state, firms' profit maximization makes their real marginal cost equal to the inverse of their markup $(\varepsilon - 1)/\varepsilon$; since real marginal cost depends positively on employment and negatively on real reserves (through borrowing costs), real reserves need to react positively to employment to keep real marginal cost equal to $(\varepsilon - 1)/\varepsilon$. The function $\mathcal{M}$ is defined over $(0, \bar{h})$, where the upper bound $\bar{h} > 0$ is the limit value of employment when real reserves tend to infinity, and we have $\lim_{h\to \bar{h}} \mathcal{M}(h) = +\infty$.

Since the steady-state price level is constant over time, households' first-order condition for bonds (4) implies that the steady-state interest rate on bonds I is equal to $1/\beta$. Using this result, as well as (3), (9), (17), (18), and (19), we rewrite households' first-order condition for reserves (7) at the steady state as

$$\mathcal{F}(h) \equiv \frac{\Gamma_m [\mathcal{L}(h), \mathcal{M}(h)]}{u'[f(h)]} = -(1 - \beta I^m). \tag{20}$$

The function $\mathcal{F}$ is defined over $(0, \bar{h})$. We show in Appendix A.5 that it is strictly increasing ($\mathcal{F}' > 0$), with $\lim_{h\to 0} \mathcal{F}(h) = -\infty$ and $\lim_{h\to \bar{h}} \mathcal{F}(h) = 0$. So, for any policy setting the IOR rate below the interest rate on bonds ($I^m < I = 1/\beta$), we have a unique steady state. When $I^m \geq I$, there is no equilibrium because banks would be tempted to issue infinite amounts of debt and deposit the proceeds at the central bank. When $I^m < I$, households' first-order condition for reserves (7) implies that the convenience yield of bank reserves is positive ($\Gamma_m < 0$), and this basically pins down the demand for real reserves. Since the nominal stock of reserves is exogenous, pinning down the demand for real reserves also pins down the price level.

Our model, thus, rules out the possibility of deflationary equilibrium paths analyzed by Benhabib et al. (2001a, 2001b). These paths are associated, in standard monetary-policy models, with an unintended deflationary steady state and a permanently binding ZLB constraint. By contrast, since the central bank sets the money supply in our model, we have a unique steady state (provided that $1 \leq I^m < 1/\beta$), and inflation is equal to the money growth rate at this steady state.[14]

Note, finally, that steady-state employment $h = \mathcal{F}^{-1}[-(1-\beta I^m)]$ is increasing in the IOR rate I^m. This is because an increase in I^m reduces the opportunity cost of holding reserves I/I^m. The lower opportunity cost increases real reserves, which in turn decreases banking costs and hence borrowing costs, which in turn stimulates employment and output.

3.2 Unique Local Equilibrium

We now consider a monetary policy setting its instruments I_t^m and M_t exogenously in the neighborhood of some constant values $I^m \in [1, 1/\beta)$ and $M > 0$. In Appendix B.1, we log-linearize the equilibrium conditions of the model around its unique steady state and get the following IS equation, Phillips curve, and reserves-demand equation:

$$\hat{y}_t = \mathbb{E}_t\{\hat{y}_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\{\pi_{t+1}\} - r_t) \tag{21}$$

$$\pi_t = \beta\mathbb{E}_t\{\pi_{t+1}\} + \kappa(\hat{y}_t - \delta_m\hat{m}_t), \tag{22}$$

$$\hat{m}_t = \chi_y\hat{y}_t - \chi_i(i_t - i_t^m), \tag{23}$$

where variables with hats denote log-deviations from steady-state values, $\pi_t \equiv \log\Pi_t$, $i_t \equiv \hat{I}_t$, $i_t^m \equiv \hat{I}_t^m$, $r_t \equiv \hat{\zeta}_t - \mathbb{E}_t\{\hat{\zeta}_{t+1}\}$, and all the parameters (σ, β, κ, δ_m, χ_y, χ_i) are positive.

The IS equation (21) is exactly the same as in the basic NK model, except that i_t is not the policy rate in our model. This IS equation directly comes from the consumption Euler equation (3)-(4) and the goods-market-clearing condition (17). The Phillips curve (22) differs from its counterpart in the basic NK model in two ways. First, it involves real reserves $\hat{m}_t$, because they reduce banking costs, which in turn lowers the borrowing costs of firms and hence their marginal cost of production. The parameter δ_m thus depends (positively) on $|\Gamma_{\ell m}|$. Second, the slope κ of the Phillips curve depends (positively) on $\Gamma_{\ell\ell}$, as an increase in output $\hat{y}_t$ raises firms' marginal cost of production also through the resulting increase in loans and banking costs. Finally, the reserves-demand equation (23) states that the demand for reserves depends positively on loans and hence output, and negatively on the marginal opportunity cost of holding reserves, measured by the spread between the IOR rate and the interest rate on bonds. The parameter χ_y thus depends positively on $|\Gamma_{\ell m}|$, and the parameter χ_i negatively on Γ_{mm}.

Under permanently exogenous monetary-policy instruments i_t^m and $\hat{M}_t$, and given the identities $\hat{m}_t = \hat{M}_t - \hat{P}_t$ and $\pi_t = \hat{P}_t - \hat{P}_{t-1}$, Equations (21)-(23) lead to a third-order dynamic equation in the price level. More specifically, this equation relates $\hat{P}_t$ to $\mathbb{E}_t\{\hat{P}_{t+2}\}$, $\mathbb{E}_t\{\hat{P}_{t+1}\}$, $\hat{P}_{t-1}$, and exogenous terms. We show in Appendix B.2 that the characteristic roots of this dynamic equation are three real numbers ρ, ω_1, and ω_2 such that $0 < \rho < 1 < \omega_1 < \omega_2$. With one

[14] Our result about steady-state uniqueness holds under an IOR-rate peg; our result about steady-state inflation equal to money growth holds more generally under any IOR-rate rule (simply because real reserves are constant over time in any steady-state equilibrium).

characteristic root inside the unit circle (ρ) for one predetermined variable ($\hat{P}_{t-1}$), thus, the dynamic equation satisfies Blanchard and Kahn's (1980) conditions and has a unique bounded solution in $\hat{P}_t$. In Appendix B.2, we solve for this unique solution and derive the implied values of inflation π_t and output $\hat{y}_t$. We get the following expression for inflation:

$$\pi_t = -(1-\rho)\,\hat{P}_{t-1} + \frac{\mathbb{E}_t}{\omega_2 - \omega_1}\left\{\sum_{k=0}^{+\infty}\left(\omega_1^{-k-1} - \omega_2^{-k-1}\right) Z_{t+k}\right\}, \tag{24}$$

where

$$Z_t \equiv \frac{-\kappa}{\beta\sigma}\left(i_t^m - r_t\right) + \left[\frac{1}{\sigma\chi_i} - \left(1 + \frac{\chi_y}{\sigma\chi_i}\right)\delta_m\right]\frac{\kappa}{\beta}\hat{M}_t + \frac{\delta_m\kappa}{\beta}\mathbb{E}_t\left\{\hat{M}_{t+1}\right\}.$$

3.3 Implications For Inflation at the ZLB

We now use the determinacy result just obtained to explain how our model, unlike standard NK models, can account for two features of US inflation during ZLB episodes: no significant deflation, and little inflation volatility.

At the ZLB, the policy rate is pegged and is therefore exogenous. In our model, permanently exogenous policy instruments deliver determinacy. In our unique equilibrium, the sum in (24) involves ω_1^{-k} and ω_2^{-k} terms with $\omega_1 > 1$ and $\omega_2 > 1$. So, the later shocks are expected to occur, the smaller their effects on current inflation. More specifically, the effects on inflation at date t of shocks occurring at date $t+k$ and announced at date t decay at an exponential rate with the horizon k.

In standard NK models, by contrast, a permanently exogenous policy rate leads to indeterminacy. Determinacy can be, and typically is, obtained by assuming that the central bank will switch, after a temporary ZLB episode, to a policy-rate rule that sets a nominal anchor (e.g., a Taylor rule like $i_t = \phi\pi_t$ with $\phi > 1$). In Diba and Loisel (2021), drawing on earlier contributions to the literature, we show that, as a result, the implied weights on anticipated future shocks like Z_{t+k} in (24) do not decay, but instead grow exponentially with the horizon k, regardless of the type of shock considered (preference, supply, monetary, fiscal, etc.).

This key difference implies that deflation can be arbitrarily large and volatile during a ZLB episode in standard NK models, but not in our model. To see this, consider, in our model, a temporary ZLB episode caused by a negative discount-factor shock between dates 0 and T ($r_t < 0$ for $0 \leq t \leq T$). We assume that cutting down the IOR rate to the ZLB only partially offsets this shock ($i_t^m - r_t = z^* > 0$ for $0 \leq t \leq T$). For simplicity, we also assume that the price level is at its steady-state value before the ZLB episode ($\hat{P}_{-1} = 0$), reserves-supply policy is neutral during this episode ($\hat{M}_t = 0$ for $0 \leq t \leq T$), and monetary policy is neutral afterwards ($i_t^m - r_t = \hat{M}_t = 0$ for $t \geq T+1$). Under these assumptions, the exogenous driving term Z_t takes the value $-\kappa z^*/(\beta\sigma)$ between dates 0 and T, and the value 0 afterwards. Therefore, we can then rewrite (24) at date 0 as

$$\pi_0 = \frac{-\kappa z^*}{\beta\sigma\left(\omega_2 - \omega_1\right)} \sum_{k=0}^{T} \left(\omega_1^{-k-1} - \omega_2^{-k-1}\right).$$

In Diba and Loisel (2021), we show that π_0 has a similar expression in the basic NK model; the key difference is that $0 < \omega_1 < 1 < \omega_2$ in the basic model, while $1 < \omega_1 < \omega_2$ in our model. So, in the basic NK model, the deflation rate $(-\pi_0)$ grows exponentially with the duration T of the ZLB episode. In our model, by contrast, it converges to the finite value $\kappa z^*/[\beta\sigma(\omega_1 - 1)(\omega_2 - 1)]$ as $T \to +\infty$.[15] Similarly, in the basic NK model, even small changes in the expected duration T of the ZLB episode can have very large effects on inflation π_0. In our model, by contrast, small changes in T will have small effects on π_0. Thus, unlike standard NK models, our model predicts no severe deflation and little inflation volatility during a temporary ZLB episode.

3.4 Robustness Analysis: Reserves-Supply Rule and Household Cash

In this section, we have shown that our benchmark model delivers local-equilibrium determinacy under exogenous monetary-policy instruments, and we have used this determinacy result to explain the low volatility of inflation and the absence of significant deflation at the ZLB. The analysis in this section rests on two simplifying assumptions: the exogeneity of nominal reserves, and the absence of household cash. In Appendix C, as a robustness check, we relax our assumption of an exogenous nominal stock of bank reserves, and consider instead a "quantitative-easing rule" setting the nominal stock of bank reserves as a function of output and the price level. We find that determinacy no longer obtains for all parameter values, but we argue that it still obtains for all *reasonable* parameter values. In Appendix D, as another robustness check, we introduce household cash into our benchmark model through a cash-in-advance constraint. We show that determinacy still obtains in the resulting model under an exogenous IOR rate and an exogenous monetary base (which is now made of bank reserves and household cash), except again for implausible calibrations.

4 Numerical Simulation of QE policies

In this section, we conduct a non-linear numerical simulation of QE policies in our benchmark model. Our main goal is to illustrate how our model, despite its monetarist features, can explain why no significant inflation was observed in the US following QE policies. More specifically, we show that large monetary expansions can have very small inflationary effects in our model if: (1) the demand for reserves is close to satiation (in the sense that I^m is close to I, or equivalently Γ_m is close to 0), and (2) the monetary expansion is perceived as temporary. To make this

[15] In Appendix B.4, we illustrate numerically these contrasting implications of the basic NK model and our model. Under our calibration of the basic NK model, the deflation rate $(-\pi_0)$ reaches large values even for relatively small expected ZLB durations, e.g. 21% per year for an expected ZLB duration of two years.

point, we first calibrate our model to a steady-state equilibrium that matches some features of the US economy in November 2010, leading up to QE2; then, we consider the effects of large monetary expansions, from one up to several times QE2.

4.1 Calibration

For the calibration, we consider iso-elastic functional forms:

$$\begin{aligned}
u(c_t) &\equiv (1-\sigma)^{-1}(c_t)^{1-\sigma}, \\
v(h_t) &\equiv V(1+\eta)^{-1}(h_t)^{1+\eta}, \\
v^b\left(h_t^b\right) &\equiv V_b(1+\eta)^{-1}\left(h_t^b\right)^{1+\eta}, \\
f(h_t) &\equiv A(h_t)^{\alpha}, \\
f^b\left(h_t^b, m_t\right) &\equiv A_b\left(h_t^b\right)^{1-\varsigma}(m_t)^{\varsigma},
\end{aligned}$$

where $\sigma > 0$, $V > 0$, $\eta \geq 0$, $V_b > 0$, $A > 0$, $0 < \alpha \leq 1$, $A_b > 0$, and $0 < \varsigma < 1$. These specifications imply

$$\begin{aligned}
g^b(\ell_t, m_t) &= A_b^{\frac{-1}{1-\varsigma}}(\ell_t)^{\frac{1}{1-\varsigma}}(m_t)^{\frac{-\varsigma}{1-\varsigma}}, \\
\Gamma(\ell_t, m_t) &= V_b(1+\eta)^{-1}A_b^{\frac{-(1+\eta)}{1-\varsigma}}(\ell_t)^{\frac{1+\eta}{1-\varsigma}}(m_t)^{\frac{-\varsigma(1+\eta)}{1-\varsigma}}.
\end{aligned}$$

We need to calibrate the parameters characterizing these functional forms (σ, V, η, V_b, A, α, A_b, ς), as well as the parameters β, ε, ϕ, θ, and I^m. However, we have three degrees of freedom in our calibration, as we can freely pick units for output y_t and labor inputs (h_t and h_t^b). So, without any loss in generality, we can set arbitrarily any three of the following four parameters: A, A_b, V, and V_b. We choose to normalize A, A_b, and V to one.

We set standard values for the parameters σ, η, α, ε, and θ that appear in standard models. The utility function is logarithmic in consumption ($\sigma = 1$) and has a unitary Frisch elasticity of labor supply, for production workers as well as bankers ($\eta = 1$). The elasticity of output with respect to the labor input is $\alpha = 0.67$. For the price-setting nexus, we set the elasticity of substitution across differentiated goods to $\varepsilon = 6$ and the Calvo price-rigidity parameter to $\theta = 0.67$ (corresponding to "three-quarter price rigidity"). In addition, we assume that firms borrow the entire wage bill ($\phi = 1$), as in Christiano et al. (2005) and Ravenna and Walsh (2006). None of these values plays a major role in our simulation results.

We set the net IOR rate $I^m - 1$ to 25 basis points per annum (the value prevailing in November 2010 in the US). The spread $I - I_m$ between the interest rate on bonds and the IOR rate plays a central role in our model. Unfortunately, this spread is not easy to calibrate because I is a shadow rate that is not directly observed (as our hypothetical bonds provide no non-pecuniary services and are in zero net supply).[16] Nagel (2016) estimates the liquidity premium

[16] Treating I_t as an unobservable shadow rate has precedents in the literature (e.g., Del Negro et al., 2017, Herrenbrueck, 2019, Geromichalos and Herrenbrueck, 2022).

on US Treasury bills using the interest rate on three-month general collateral (GC) repurchase agreements as the rate on a risk-free but illiquid asset.[17] The (gross) GC repo rate I_{GC} is not a suitable measure of our shadow rate I because the former can reflect a convenience yield, unlike the latter. Specifically, participating banks are typically lenders in the GC repo transaction and benefit from borrowing the securities used as collateral (as this gives them the certainty of supply of collateral for three months). So, denoting the convenience yield of borrowed Treasury securities by C_b, the convenience yield of holding Treasury bills outright by C_o, and the (gross) T-bill yield by I_{TB}, we have

$$I - I_{GC} = C_b, \quad I - I_{TB} = C_o, \quad \text{and} \quad I_{GC} - I_{TB} = C_o - C_b.$$

The repo-T-bill spread $I_{GC} - I_{TB}$ (reported on Stefan Nagel's website) stood at 8 basis points per annum in November 2010. If we assume, for example, that the convenience yield of borrowed Treasury securities C_b is half as large as the convenience yield of T-bills held outright C_o, we get a value of 16 basis points per annum for C_o. Adding the average net T-bill yield in November 2010 (14 basis points per annum) to this estimate of the $I - I_{TB}$ spread, we get a net shadow rate $I - 1$ of 30 basis points per annum. In this case, our estimate for the $I - I_m$ spread would be 5 basis points per annum. Alternatively, we could get larger (smaller) estimates of the $I - I_m$ spread by assuming that the convenience yield of borrowed Treasury securities C_b is more (less) than half as large as the convenience yield of T-bills held outright C_o. As we will see below, smaller $I - I_m$ spreads are more favorable to our main point in this section. To err on the conservative side, thus, we set our benchmark value for the $I - I_m$ spread to 10 basis points per annum, and we consider alternative values of 5 and 20 basis points per annum. Since we have no inflation in the steady state, our benchmark value for the net shadow rate $I - 1$ (35 basis points per annum) pins down the discount factor to $\beta = 1/I = 0.999$ on a quarterly basis.

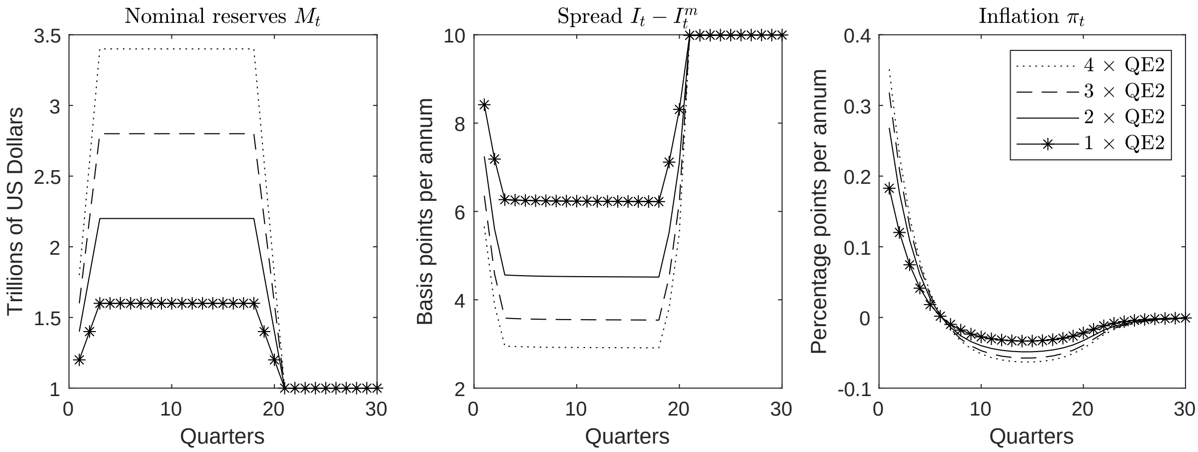
We set the remaining two parameters, ς and V_b, so as to reach the following two steady-state targets: (i) the net interest rate on bank loans $I^{\ell} - 1$ is 3.25% per annum (the prime loan rate in November 2010 in the US); and (ii) the ratio of bank reserves to loans is $m/\ell = 1/9$ (the ratio of total reserves to bank credit of all commercial banks in November 2010 in the US). In Appendix A.6, we show how these targets pin down ς and V_b; we get $\varsigma = 0.0039$ and $V_b = 0.019$. Our simulation results, reported and discussed in the next subsection, are not sensitive to plausible variations in the values we pick for these targets.

4.2 Simulations

To assess the quantitative effects of large monetary expansions, we need to work with the non-linear version of our model. We use the "simul" command of Dynare for our non-linear simulation

[17] The associated loans are risk-free because they are fully collateralized with Treasury securities; they have very little liquidity, although one side of the transaction (or both sides) may have an option to terminate it earlier.

Figure 1 – Effects of temporary balance-sheet expansions



Note: The figure displays the effects of announcing at date 1 a temporary balance-sheet expansion (left panel) on the spread (middle panel) and inflation (right panel) between dates 1 and 30.

of a perfect-foresight equilibrium that asymptotically converges to the steady-state equilibrium. Figure 1 shows the effects of four alternative monetary expansions. One, like QE2, raises the balance-sheet size from an already large value ($1 trillion) to a substantially larger one ($1.6 trillion) in the course of 3 quarters (solid line with asterisks in Figure 1). The others raise the balance-sheet size by two, three, or four times as much, i.e. from $1 to $2.2, $2.8, or $3.4 trillion (solid, dashed, and dotted lines in Figure 1). All these monetary expansions are temporary: the balance-sheet size rises over 3 quarters, remains at its new value for 15 quarters, and goes back to its initial value over 3 quarters.

As shown in Figure 1, the "single QE2" expansion makes the $I_t - I_t^m$ spread fall from 10 to 6.2 basis points per annum, and raises annualized inflation by only 18 basis points upon impact. And the "multiple QE2" expansions do not have much larger inflationary effects: following the "double, triple, and quadruple QE2" expansions, the spread falls to 4.5, 3.5, and 2.9 basis points, and inflation rises by only 27, 32, and 35 basis points respectively. These results illustrate the strongly decreasing returns to scale of quantitative easing in our setup. These strongly decreasing returns to scale are also apparent in Table 1 below, which extends the results to still larger expansions (up to 32 times QE2).

Our results are not sensitive to the values we assume for most of our parameters (although we could make the impact effects on inflation even smaller if we raised the price-rigidity parameter, say, to $\theta = 0.75$). Only two features really matter for the results.

First, our simulations start with a small spread $I_t - I_t^m$. This reflects a presumption that the already large level of reserve balances in the US prior to QE2 had "nearly satiated" the demand for real reserves (in the sense of bringing Γ_m close to 0). In this case, as the reserves-demand equation (7) makes clear, a large increase in nominal-reserves supply M_t can be absorbed by a small drop in the spread $I_t - I_t^m$, without changing the price level P_t by much. If we set the steady-state spread $I - I^m$ to 5 basis points (instead of 10), we get even closer to satiation of demand for reserves, and the inflationary impact of our "single QE2" expansion drops from 18 to 9 basis points. Conversely, if we moved further away from satiation and set $I - I^m$ to 20 basis points (instead of 10), the inflationary impact of our "single QE2" expansion would rise from 18 to 37 basis points. In short, the inflationary impact moves roughly one-for-one with the steady-state spread for plausible values of the latter.

The second assumption that matters for our low-inflation result is that the balance-sheet expansion is expected to be temporary. To see why this assumption matters, note that in the extreme case of a permanent increase in nominal reserves, our model would imply a proportional price increase in the long term. The reason is that the central bank does not change I^m in our QE experiment, and our representative-consumer setup pins down $I = 1/\beta$; so, the steady-state spread $I - I^m$ cannot shrink to raise the demand for real reserves if our monetary expansion is permanent; and the price level has to increase eventually by the same factor as the nominal stock of reserves, so as to leave the steady-state real reserve balances unchanged. To offset the long-term effect of a permanent increase in the supply of reserves on the price level, the central bank would have to raise the IOR rate in order to stimulate demand for reserves.

Our assumption that, as of 2010, the unusual monetary expansion was not expected to last more than 5 years does not seem unreasonable to us, in light of commentary on how the crisis was not expected to last as long as it did. At any rate, the inflationary effects of temporary monetary expansions that are expected to last reasonably longer than 5 years are also modest in our model. Table 1 shows the results for expansions that are expected to last 10 or 20 years (instead of 5). For our "single QE2" expansion, extending the duration from 5 to 10 years raises the impact effect on annualized inflation from 18 to 40 basis points. Although our numerical simulations are non-linear, the log-linear approximation (24) of the solution for the inflation rate can offer a mechanical explanation for this result. Our calibration makes the root ω_1 very close to 1 (equal to 1.0003), and the root ω_2 substantially larger (equal to 1.42); so, the factor $\omega_1^{-k-1} - \omega_2^{-k-1}$ on the right-hand-side of (24) is close to 1 for a large range of horizons k.[18] Doubling the expected duration of the QE experiment from 5 to 10 years doubles the number of non-zero terms on the right-hand-side of (24); given that the relevant "discount factor" is close to 1, this roughly doubles the inflationary impact. As Table 1 illustrates, the response of the inflationary impact to the expected duration of the expansion remains approximately one-for-one at least for expansion sizes up to 32 times QE2 and for expansion durations up to 20 years. Thus, forward guidance about the duration of the expansion is a powerful tool to control inflation in our calibrated model.

In retrospect, the QE2 expansion was not reversed in 10 years, and we saw much larger balance-sheet expansions during QE3 and in response to the COVID crisis. The Fed's balance sheet has

[18]This result holds over the range of plausible values for the steady-state spread $I - I^m$: ω_1 moves from 1.0001 to 1.0005 – and thus remains very close to 1 – as the spread moves from 5 to 20 basis points.

Table 1 – Inflationary impact (in basis points per annum) of balance-sheet expansions

	5 years	10 years	20 years
$1 \times$ QE2	18	40	84
$2 \times$ QE2	27	59	122
$4 \times$ QE2	35	76	157
$8 \times$ QE2	42	90	184
$16 \times$ QE2	46	98	202
$32 \times$ QE2	48	104	212

Note: the table displays the increase in inflation, expressed in basis points per annum, at the start of an unexpected balance-sheet expansion of size 1 to 32 times QE2 and of duration 5, 10, or 20 years.

grown to about $9 trillion during this process. What insights does our model offer about the inflationary effects of QE? We discuss our results and express our views (which admittedly go beyond the formal results presented in this paper) in the following subsection.

4.3 Discussion

One way to reconcile large monetary expansions with low inflation, in the aftermath of the Great Recession, is to assume that demand for money was satiated, and total public-sector liabilities were not expected to grow rapidly.[19] Our goal, however, is to account for low inflation while preserving local-equilibrium determinacy. In contrast to a model with full satiation of money demand, our model views inflation through the lens of the supply and demand for money. A permanent increase in the money supply is inflationary, according to our model, unless it is accompanied by an increase in money demand.

Our simulations, however, suggest that transitory increases in the money supply may have small inflationary effects when the spread between the IOR rate I_m and the (unobservable) shadow rate I is small.[20] Under our iso-elastic functional forms, the semi-elasticity of demand for reserves is inversely related to this spread. So, our simulations suggest that the inflationary effects of QE are small when the semi-elasticity of demand for reserves is large (i.e. when the function Γ_m in our model is fairly flat). In this case, small changes in the spread (the opportunity cost of holding reserves) can absorb large changes in the supply of reserves. The fact that the relevant spread is not observable is a limitation of our setup. As we noted, however, our simulations support our claims for spread values between 5 and 20 basis points per annum.

Our simulations start with the large balance sheet prevailing just before QE2. We don't think our model can explain why the QE1 expansion was not inflationary: it seems hard to argue that

[19] A number of earlier contributions (e.g., Andolfatto, 2015) present models in which inflation is pinned down by the growth rate of total public-sector liabilities, when the interest rates on money and bonds are equalized.

[20] The key role played by this shadow rate in our model is not a consequence of our assumption that households have access to bonds providing no non-pecuniary services and being in zero net supply. The exact same shadow rate would appear in Equation (7) even if we assumed that households do not have access to such bonds.

the function Γ_m was flat starting with reserve balances around \$45 billion, as was the case in 2008 just before QE1. Our view is that the QE1 expansion was primarily decided in response to, and largely absorbed by, an increase in demand for bank reserves. When the federal-funds market collapsed, and the Fed introduced interest payment on reserves in October 2008, holding reserve balances at the Fed largely replaced transactions in the interbank market and the use of inside assets (like commercial paper) for liquidity management. Subsequently, the liquidity-coverage requirements associated with Basel III and the Fed's stress tests substantially increased the demand for reserves by US banks. This suggests that part of the balance-sheet expansions would eventually not raise the price level even if they came to be viewed as permanent.

Our results do not imply that large and repeated balance-sheet expansions are always benign. Balance-sheet expansions can generate larger inflationary pressures in our setup if they are not expected to be reversed anytime soon or if the $I - I^m$ spread is large. Viewed through the lens of our model, it seems possible that an increase in the expected duration of balance-sheet expansions contributed to the surge in US inflation in the aftermath of the COVID crisis; and that a higher value of expected inflation (due to adverse supply shocks) made these balance-sheet expansions more inflationary via the Fisher effect raising I, for a given (constant) value of I^m. Our observations suggest that in the aftermath of large and persistent balance-sheet expansions, central banks should be mindful of changes in the expected duration of these expansions, or changes in economic conditions that can affect the shadow rate I, as these changes can have sizeable effects on inflation. To stabilize inflation, central banks may respond to these changes by communicating about the intended duration of the expansions, or by modifying the IOR rate in order to adjust the demand for reserves.

5 Extension With Liquid Government Bonds

In the preceding two sections, we have shown that our benchmark model can broadly account for three key observations about US inflation during ZLB episodes (no significant deflation, little inflation volatility, and no significant inflation following QE policies). These results rest on the assumption that demand for bank reserves got close to satiation, but did not reach full satiation, i.e. that bank reserves carried a small but positive convenience yield ($I_t^m < I_t$ and $\Gamma_m > 0$).

In Appendix E, we address an important argument that goes against our non-satiation view. This argument is the fact that T-bill returns were often below the IOR rate during ZLB episodes. We do not think this fact contradicts our claim that reserves still had a positive marginal convenience yield during these episodes. The lower T-bill returns, we argue, could reflect strong demand by non-bank entities – using T-bills as collateral or international reserve asset, for instance. We formalize our counter-argument by introducing government bonds providing liquidity services into our benchmark model. We show that our model with liquid bonds has an equilibrium in which the return on government bonds is below the IOR rate, while demand for bank reserves is not satiated. Moreover, this equilibrium of our model with liquid bonds coincides with the equilibrium of our benchmark model (without liquid bonds), in the sense that all the endogenous variables that are common to both models, except the lump-sum transfer T_t, take the same equilibrium values. So, all the results that we have obtained in our benchmark model in Sections 3 and 4 still hold in our model with liquid bonds. Thus, our model with liquid bonds can account not only for the negative spread between T-bill and IOR rates observed during ZLB episodes, but also for the three key features of inflation during these episodes.

6 Normalization and Operational Framework of Monetary Policy

In the preceding sections, we have used our model to explain some key observations about inflation and money-market rates during ZLB episodes. In the present section, we now explore the implications of our model for the normalization and the operational framework of monetary policy.

6.1 Normalization of Monetary Policy

The issue of monetary-policy normalization, i.e. raising policy rates from the ZLB and shrinking the size of the central bank's balance sheet, has received considerable attention since the 2008-2015 ZLB episode. After this episode, the Fed started to normalize its monetary policy, raising the IOR rate from December 2015 to December 2018 and reducing the size of its balance sheet from October 2017 to August 2019, before reversing course due to the COVID crisis. Since March 2022, the Fed has raised again the IOR rate, in a high-inflation context.

Our model provides a simple framework to think about the effects of normalizing monetary policy. In particular, our model has unambiguous implications about the effects of normalizing monetary policy on inflation. More specifically, in our log-linearized model under exogenous monetary-policy instruments (studied in Section 3), current and expected future IOR-rate hikes and balance-sheet contractions always exert deflationary pressures. To establish this result, we use the definition of the exogenous driving term Z_t (as a function of $i_t^m - r_t$, $\hat{M}_t$, and $\hat{M}_{t+1}$) to rewrite (24) as

$$\begin{aligned}\pi_t \quad = \quad & -(1-\rho)\,\hat{P}_{t-1} + \frac{(1-\delta_m\chi_y)\,\kappa}{\beta\sigma\chi_i\,(\omega_1-1)\,(\omega_2-1)}\hat{M}_{t-1} \\ & + \frac{\kappa}{\beta\,(\omega_2-\omega_1)}\mathbb{E}_t\left\{-\frac{1}{\sigma}\sum_{k=0}^{+\infty}\left(\omega_1^{-k-1}-\omega_2^{-k-1}\right)\left(i_{t+k}^m - r_{t+k}\right)\right. \\ & \left. + \sum_{k=0}^{+\infty}\left[\left(\frac{1-\delta_m\chi_y}{\sigma\chi_i}\right)\left(\frac{\omega_1^{-k}}{\omega_1-1}-\frac{\omega_2^{-k}}{\omega_2-1}\right)+\delta_m\left(\omega_1^{-k}-\omega_2^{-k}\right)\right]\hat{\mu}_{t+k}\right\}, \qquad (25)\end{aligned}$$

where $\hat{\mu}_t = \hat{M}_t - \hat{M}_{t-1}$ denotes the log-deviation of the gross growth rate of nominal reserves $\mu_t \equiv M_t/M_{t-1}$ from its steady-state value 1. Since $\omega_2 > \omega_1 > 1$ and $\delta_m\chi_y < 1$ (as shown in

Appendix B.3), the coefficient of i^m_{t+k} in (25) is negative, and the coefficient of $\hat{\mu}_{t+k}$ is positive. Therefore, announcing at date t a positive i^m_{t+k} or a negative $\hat{\mu}_{t+k}$, for any $k \geq 0$, lowers current inflation π_t.

So, in particular, our model does not share the Neo-Fisherian implication of some equilibria in NK models − discussed in Schmitt-Grohé and Uribe (2017) and Bilbiie (2022), among others − that suggest interest-rate hikes may serve to raise inflation to target in an environment with deflationary pressures. Our result, of course, is about IOR-rate hikes, rather than hikes in the interest rate appearing in the IS equation of the NK models analyzed in the literature.

Note that we have obtained this result (that monetary-policy normalization always has deflationary effects) only because the "unstable eigenvalues" of the dynamic system, ω_1 and ω_2, are always positive real numbers. If ω_1 and ω_2 had instead been negative real numbers or conjugate complex numbers, then inflation would still have been characterized by (25), but the sign of the coefficients of i^m_{t+k} and $\hat{\mu}_{t+k}$ in (25) would have then depended on the horizon k. Expected future IOR-rate hikes would have been deflationary for some hike horizons, and inflationary for others; and similarly for expected future balance-sheet contractions. In Diba and Loisel (2021), we show that other, less structured monetary models, in particular the familiar MIU model with separable or non-separable utility, allow for conjugate complex eigenvalues, unlike our model. Thus, the additional structure brought by our model is key to obtain our unambiguous result about the deflationary effects of monetary-policy normalization.

6.2 Floor System

How will monetary policy be conducted away from the ZLB, after it is normalized? Over the past few years, the Fed has repeatedly stated its intention to keep the balance sheet large (or let it shrink slowly and predictably over time as central-bank assets mature), without actively managing the quantity of reserves, and to set the interest rate on reserves (and, perhaps, the reverse-repo rate) depending on the state of the economy (see, e.g., Federal Open Market Committee, 2019). This operational framework is often referred to as a "floor system."

In this subsection, we investigate the consequences of a floor system for local-equilibrium determinacy in our model. We consider a floor system in which policy sets the size of the balance sheet exogenously and sets the IOR rate depending on the state of the economy. We show that if the IOR rate reacts only to current inflation, then we get determinacy for any non-negative response, in contrast to what we get in standard NK models; and if the IOR rate also reacts to current output, determinacy conditions remain quite lax.

More specifically, we consider a floor system under which the central bank sets the stock of nominal reserves M_t exogenously (around a constant value $M > 0$, as previously) and sets the

IOR rate I_t^m according to the Taylor rule

$$I_t^m = \mathcal{R}\left(\Pi_t, y_t\right),$$

where the function $\mathcal{R}$, from $\mathbb{R}^2_{>0}$ to $[1, +\infty)$, is differentiable and non-decreasing in Π_t and y_t (i.e. $\mathcal{R}_\Pi \geq 0$ and $\mathcal{R}_y \geq 0$). Under this floor system, the set of steady states is characterized by

$$\mathcal{F}\left(h\right) = -\left\{1 - \beta\mathcal{R}\left[1, f\left(h\right)\right]\right\},$$

which corresponds to (20) with I^m replaced by $\mathcal{R}[1, f(h)]$. Given the properties of the function $\mathcal{F}$, a sufficient condition for existence of a steady state (which is also a necessary condition for existence and uniqueness of a steady state) is

$$\mathcal{R}\left[1, f\left(\bar{h}\right)\right] < \frac{1}{\beta}.$$

Log-linearizing the model around a steady state, we get the same IS equation (21), Phillips curve (22), and reserves-demand equation (23) as previously, plus now the Taylor rule

$$i_t^m = r_\pi \pi_t + r_y \hat{y}_t, \tag{26}$$

where $r_\pi \equiv \mathcal{R}_\Pi / \mathcal{R} \geq 0$ and $r_y \equiv (\mathcal{R}_y y)/\mathcal{R} \geq 0$.

In Appendix B.5, we derive the necessary and sufficient condition for local-equilibrium determinacy under this floor system. We show in particular that a sufficient condition is

$$r_y < \frac{1 - \delta_m \chi_y}{\delta_m \chi_i}, \tag{27}$$

where the right-hand side is positive. If the IOR rate reacts only to inflation (i.e. $r_y = 0$), then Condition (27) is necessarily met. In this case, determinacy obtains for any non-negative reaction to inflation (i.e. for any value of $r_\pi \geq 0$), and the Taylor principle does not apply. Alternatively, if the IOR rate reacts also to output (i.e. $r_y > 0$), then, to get a sense of how lax or stringent Condition (27) is, we consider the same calibration as in Subsection 4.1, which involves a large balance sheet (consistently with the discussion of a floor system for the Fed). Under this calibration, we get the value 15.7 for the right-hand side of (27). This threshold value seems comfortably high, given that the Taylor-rule coefficient on output is typically one order of magnitude lower in the literature. Thus, we view Condition (27) as likely to be met, and therefore determinacy as likely to prevail, under such a floor system in our model.

While our model delivers determinacy under both passive and active interest-rate rules, alternative rules do differ in our setup – in terms of their stabilization properties. In Appendix B.6, motivated in part by the recent (2021-2022) surge in US inflation, we add an adverse supply shock to our model and compare the performance of alternative rules that link the policy rate to inflation ("performance" of the rules in terms of their ability to stabilize inflation following the shock). Interestingly, the performance of policy rules *deteriorates* as we raise the response of the policy rate to inflation over the [0,1] interval, and then *improves* as the rules turn more active (as we raise the response coefficient above unity). Our discussion in Appendix B.6 relates these numerical results to the cost channel of monetary policy.

7 Conclusion

Central banks conduct monetary policy by setting, ultimately, two instruments that they directly control: the interest rate on bank reserves and the size of their balance sheet. In this paper, we have taken this fact seriously and proposed a model in which the central bank sets these two instruments. Central-bank liabilities consist only of bank reserves in the benchmark version of the model, but include also household cash in an extended version.

We show that the model can account, in qualitative terms, for the three key observations made by Cochrane (2018) about US inflation at the ZLB: no significant deflation, little inflation volatility, and no significant inflation following quantitative-easing policies. In addition, we show that with liquid government bonds, the model can also account for the negative spread between Treasury-bill and IOR rates observed during this ZLB episode. We analyze the implications of our model for the normalization and operational framework of monetary policy. On this front, we do not find any possibility of Neo-Fisherian effects during the monetary-policy normalization process, as current and expected future IOR-rate hikes and balance-sheet contractions always exert deflationary pressures in our model. And we show that under a floor system, the central bank need not follow an active IOR-rate rule in order to ensure local-equilibrium determinacy: a passive IOR-rate rule (even an IOR-rate peg) works as well.

As we noted in the Introduction, our model with near satiation of demand for reserves is not the only way one can address our broad observations about the Great Recession. Models with full satiation of money demand – and, thus, no inflationary effects of QE – can provide an alternative if they set a nominal anchor. Prominent examples in the literature are the Fiscal Theory as well as models that depart from rational expectations and deliver determinacy under an interest-rate peg. Unlike these models, our model implies that large balance-sheet expansions can have inflationary aftermaths. This is likely to be the case if the expansions come to be viewed as permanent or long-lasting, all the more so if the opportunity cost of holding reserves starts to rise.

The opportunity cost of holding reserves, which plays a key role in our model, is measured by the spread between a shadow rate and the IOR rate (independently of our simplifying assumption that households have access to hypothetical bonds providing no non-pecuniary services and being in zero net supply). The fact that this shadow rate is not observable is a limitation for our model's quantitative implications about the inflationary effects of QE. In future work, we hope to develop a model with a role for observable spreads; such a model will require more detail about the banking sector and the demand for reserves and other assets.

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Online Appendix to
"A Model of Post-2008 Monetary Policy"

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In this Online Appendix, to lighten up the notation, we sometimes omit function arguments when no ambiguity results.

Appendix A: Non-Linear Benchmark Model

In this appendix, we prove the existence of the function $\mathcal{M}$, and we establish the properties of the functions f^b, g^b, Γ, $\mathcal{M}$, and $\mathcal{F}$ (which play a role in our benchmark model). We also show how our steady-state targets pin down the parameters in the calibration of our benchmark model under iso-elastic production and utility functions. We omit time subscripts to lighten up the notation, thus writing h^b, ℓ, and m instead of h^b_t, ℓ_t, and m_t.

A.1 Concavity of the Function f^b

Since f^b is homogeneous of degree d, we have $\forall x \in \mathbb{R}_{\geq 0}$, $f^b(xh^b, xm) = x^d f^b(h^b, m)$. Computing the first derivative of the left- and right-hand sides of this equation with respect to x at $x = 1$ leads to

$$df^b = h^b f^b_h + m f^b_m. \tag{A.1}$$

In turn, computing the first derivative of the left- and right-hand sides of the last equation with respect to h^b and m leads to

$$f^b_{hh} = -\frac{(1-d)\, f^b_h + m f^b_{hm}}{h^b} \quad \text{and} \quad f^b_{mm} = -\frac{(1-d)\, f^b_m + h^b f^b_{hm}}{m}.$$

Using these expression for f^b_{hh} and f^b_{hm}, as well as (A.1), we get

$$\begin{aligned} f^b_{hh} f^b_{mm} - \left(f^b_{hm}\right)^2 &= \frac{1-d}{h^b m}\left[(1-d)\, f^b_h f^b_m + f^b_{hm}\left(h^b f^b_h + m f^b_m\right)\right] \\ &= \frac{1-d}{h^b m}\left[(1-d)\, f^b_h f^b_m + df^b f^b_{hm}\right] \\ &\geq 0, \end{aligned}$$

which implies (together with $f^b_{hh} \leq 0$ and $f^b_{mm} \leq 0$) that the function f^b is (weakly) concave.

A.2 Properties of the Function g^b

Computing the first and second derivatives of the left- and right-hand sides of $\ell = f^b[g^b(\ell, m), m]$ with respect to ℓ and m gives

$$1 = f_h^b g_\ell^b, \quad 0 = f_h^b g_m^b + f_m^b, \quad 0 = f_{hh}^b \left(g_\ell^b\right)^2 + f_h^b g_{\ell\ell}^b,$$
$$0 = f_{hh}^b g_\ell^b g_m^b + f_{hm}^b g_\ell^b + f_h^b g_{\ell m}^b, \quad \text{and} \quad 0 = f_{hh}^b \left(g_m^b\right)^2 + 2 f_{hm}^b g_m^b + f_h^b g_{mm}^b + f_{mm}^b.$$

Using these equations and $f_h^b > 0$, $f_m^b > 0$, $f_{hh}^b < 0$, $f_{hm}^b \geq 0$, and $f_{mm}^b < 0$, we sequentially get

$$g_\ell^b = \frac{1}{f_h^b} > 0, \quad g_m^b = \frac{-f_m^b}{f_h^b} < 0, \quad g_{\ell\ell}^b = \frac{-f_{hh}^b}{\left(f_h^b\right)^3} > 0,$$
$$g_{\ell m}^b = \frac{f_m^b f_{hh}^b}{\left(f_h^b\right)^3} - \frac{f_{hm}^b}{\left(f_h^b\right)^2} < 0, \quad \text{and} \quad g_{mm}^b = \frac{-f_{hh}^b \left(f_m^b\right)^2}{\left(f_h^b\right)^3} + 2\frac{f_m^b f_{hm}^b}{\left(f_h^b\right)^2} - \frac{f_{mm}^b}{f_h^b} > 0.$$

Then, using these expressions for $g_{\ell\ell}^b$, g_{mm}^b, $g_{\ell m}^b$, and the concavity of f^b, we easily get

$$g_{\ell\ell}^b g_{mm}^b - \left(g_{\ell m}^b\right)^2 = \frac{f_{hh}^b f_{mm}^b - \left(f_{hm}^b\right)^2}{\left(f_h^b\right)^4} \geq 0,$$

which implies (together with $g_{\ell\ell}^b > 0$ and $g_{mm}^b > 0$) that the function g^b is (weakly) convex.

Moreover, since f^b is homogeneous of degree d, we have $\forall x \in \mathbb{R}_{\geq 0}$, $g^b(x^d \ell, xm) = x g^b(\ell, m)$. Computing the first derivative of the left- and right-hand sides of this equation with respect to x at $x = 1$ leads to

$$g^b = d\ell g_\ell^b + m g_m^b. \tag{A.2}$$

In turn, computing the first derivative of the left- and right-hand sides of the last equation with respect to ℓ and m leads to

$$g_{\ell\ell}^b = \frac{(1-d)\, g_\ell^b - m g_{\ell m}^b}{d\ell}, \tag{A.3}$$
$$g_{mm}^b = \frac{-d\ell g_{\ell m}^b}{m}. \tag{A.4}$$

Finally, as a direct consequence of $\lim_{m\to+\infty} f_m^b(h^b, m) = 0$ and $\lim_{m\to 0} f_h^b(h^b, m) = 0$, we get $\lim_{m\to+\infty} g_m^b(\ell, m) = 0$ and $\lim_{m\to 0} g_\ell^b(\ell, m) = +\infty$ for all $\ell \in \mathbb{R}_{\geq 0}$.

A.3 Properties of the Function Γ

Computing the first and second derivatives of the left- and right-hand sides of $\Gamma(\ell, m) \equiv v^b[g^b(\ell, m)]$ with respect to ℓ and m gives

$$\Gamma_\ell = v^{b\prime} g_\ell^b > 0, \quad \Gamma_m = v^{b\prime} g_m^b < 0, \quad \Gamma_{\ell\ell} = v^{b\prime\prime} \left(g_\ell^b\right)^2 + v^{b\prime} g_{\ell\ell}^b > 0,$$
$$\Gamma_{\ell m} = v^{b\prime\prime} g_\ell^b g_m^b + v^{b\prime} g_{\ell m}^b < 0, \quad \text{and} \quad \Gamma_{mm} = v^{b\prime\prime} \left(g_m^b\right)^2 + v^{b\prime} g_{mm}^b > 0,$$

where the inequalities follow from $v^{b\prime} > 0$, $v^{b\prime\prime} \geq 0$, $g^b_\ell > 0$, $g^b_m < 0$, $g^b_{\ell\ell} > 0$, $g^b_{mm} > 0$, and $g^b_{\ell m} < 0$. In addition, using first (A.3)-(A.4) and then (A.2), we easily get

$$\begin{aligned}
\Gamma_{\ell\ell}\Gamma_{mm} - (\Gamma_{\ell m})^2 &= \left(v^{b\prime}\right)^2 \left[g^b_{\ell\ell}g^b_{mm} - \left(g^b_{\ell m}\right)^2\right] \\
&\quad + v^{b\prime}v^{b\prime\prime}\left[\left(g^b_\ell\right)^2 g^b_{mm} + \left(g^b_m\right)^2 g^b_{\ell\ell} - 2g^b_\ell g^b_m g^b_{\ell m}\right] \\
&= \frac{-(1-d)\left(v^{b\prime}\right)^2 g^b_\ell g^b_{\ell m}}{m} \\
&\quad + \frac{v^{b\prime}v^{b\prime\prime}}{d\ell m}\left[-g^b_{\ell m}\left(d\ell g^b_\ell + m g^b_m\right)^2 + (1-d)\, m g^b_\ell \left(g^b_m\right)^2\right] \\
&= \frac{-(1-d)\left(v^{b\prime}\right)^2 g^b_\ell g^b_{\ell m}}{m} \\
&\quad + \frac{v^{b\prime}v^{b\prime\prime}}{d\ell m}\left[-\left(g^b\right)^2 g^b_{\ell m} + (1-d)\, m g^b_\ell \left(g^b_m\right)^2\right] \\
&\geq 0, \qquad\qquad \text{(A.5)}
\end{aligned}$$

which implies (together with $\Gamma_{\ell\ell} > 0$ and $\Gamma_{mm} > 0$) that the function Γ is (weakly) convex. Finally, as a direct consequence of $\lim_{m\to+\infty} g^b_m(\ell, m) = 0$ and $\lim_{m\to 0} g^b_\ell(\ell, m) = +\infty$, we get $\lim_{m\to+\infty} \Gamma_m(\ell, m) = 0$ and $\lim_{m\to 0} \Gamma_\ell(\ell, m) = +\infty$ for all $\ell \in \mathbb{R}_{\geq 0}$.

A.4 Existence and Properties of the Function $\mathcal{M}$

Using (3), (5), (9), (13), (17), and (18), we can rewrite households' first-order condition for loans (6), at the steady state, as a relationship between reserves m and employment h:

$$\Gamma_\ell\left[\mathcal{L}(h), m\right] = \mathcal{A}(h) \equiv \frac{u'\left[f(h)\right]}{\phi}\left\{\left(\frac{\varepsilon - 1}{\varepsilon}\right)\frac{u'\left[f(h)\right] f'(h)}{v'(h)} - 1\right\}. \qquad \text{(A.6)}$$

Because the left-hand side of (A.6) is positive, we restrict the domain of the function $\mathcal{A}$ to $(0, h^\star)$, where $h^\star > 0$ is implicitly and uniquely defined by $u'[f(h^\star)]f'(h^\star)/v'(h^\star) = \varepsilon/(\varepsilon - 1)$. The value $h^\star$ is the value that h would take in the absence of financial frictions, i.e. if the marginal banking cost Γ_ℓ were zero. The function $\mathcal{A}$ is strictly decreasing ($\mathcal{A}' < 0$), with $\lim_{h\to 0} \mathcal{A}(h) = +\infty$ and $\lim_{h\to h^\star} \mathcal{A}(h) = 0$.

Since $\Gamma_{\ell\ell} > 0$, $\mathcal{L}' > 0$, $\Gamma_{\ell m} < 0$, and $\mathcal{A}' < 0$, Equation (A.6) implicitly and uniquely defines a function $\mathcal{M}$ such that

$$m = \mathcal{M}(h).$$

This function is strictly increasing ($\mathcal{M}' > 0$). Moreover, since $\lim_{m\to 0} \Gamma_\ell(\ell, m) = +\infty$, $\mathcal{M}$ is defined over $(0, \bar{h})$, where $\bar{h} \in (0, h^\star]$ is implicitly and uniquely defined by $\lim_{m\to+\infty} \Gamma_\ell[\mathcal{L}(\bar{h}), m] =$

$\mathcal{A}(\bar{h})$.[1] Finally, this last equation straightforwardly implies

$$\lim_{h\to\bar{h}} \mathcal{M}(h) = +\infty. \tag{A.7}$$

A.5 Properties of the Function $\mathcal{F}$

Using (A.6), we can rewrite $\mathcal{F}(h)$ as

$$\mathcal{F}(h) = \frac{1}{\phi}\mathcal{F}_1(h)\,\mathcal{F}_2(h),$$

where the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ are defined over $(0, \bar{h})$ by

$$\begin{aligned} \mathcal{F}_1(h) &\equiv \frac{\Gamma_m[\mathcal{L}(h), \mathcal{M}(h)]}{\Gamma_\ell[\mathcal{L}(h), \mathcal{M}(h)]} = \frac{g_m^b[\mathcal{L}(h), \mathcal{M}(h)]}{g_\ell^b[\mathcal{L}(h), \mathcal{M}(h)]}, \\ \mathcal{F}_2(h) &\equiv \left(\frac{\varepsilon-1}{\varepsilon}\right)\frac{u'[f(h)]\,f'(h)}{v'(h)} - 1. \end{aligned}$$

We have

$$\begin{aligned} \left(g_\ell^b\right)^2 \mathcal{F}_1' &= g_\ell^b\left(g_{\ell m}^b \mathcal{L}' + g_{mm}^b \mathcal{M}'\right) - g_m^b\left(g_{\ell\ell}^b \mathcal{L}' + g_{\ell m}^b \mathcal{M}'\right) \\ &= -g_{\ell m}^b\left(d\mathcal{L}g_\ell^b + \mathcal{M}g_m^b\right)\left(\frac{\mathcal{M}'}{\mathcal{M}} - \frac{\mathcal{L}'}{d\mathcal{L}}\right) - (1-d)\,g_\ell^b g_m^b \frac{\mathcal{L}'}{d\mathcal{L}} \\ &= -g^b g_{\ell m}^b\left(\frac{\mathcal{M}'}{\mathcal{M}} - \frac{\mathcal{L}'}{d\mathcal{L}}\right) - (1-d)\,g_\ell^b g_m^b \frac{\mathcal{L}'}{d\mathcal{L}}, \end{aligned}$$

where the second equality follows from (A.3)-(A.4), and the third equality from (A.2). Now, deriving the left- and right-hand sides of (A.6) with respect to h gives

$$\Gamma_{\ell m}\mathcal{M}' + \Gamma_{\ell\ell}\mathcal{L}' = \mathcal{A}' < 0.$$

Moreover, using (A.2) and (A.3), we get

$$\begin{aligned} d\mathcal{L}\Gamma_{\ell\ell} + \mathcal{M}\Gamma_{\ell m} &= d\mathcal{L}\left[v^{b\prime\prime}\left(g_\ell^b\right)^2 + v^{b\prime} g_{\ell\ell}^b\right] + \mathcal{M}\left(v^{b\prime\prime} g_\ell^b g_m^b + v^{b\prime} g_{\ell m}^b\right) \\ &= v^{b\prime\prime} g_\ell^b\left(d\mathcal{L}g_\ell^b + \mathcal{M}g_m^b\right) + v^{b\prime}\left(d\mathcal{L}g_{\ell\ell}^b + \mathcal{M}g_{\ell m}^b\right) \\ &= v^{b\prime\prime} g^b g_\ell^b + (1-d)\,v^{b\prime} g_\ell^b \\ &\geq 0. \end{aligned}$$

The last two inequalities together imply

$$\frac{\mathcal{M}'}{\mathcal{M}} > \frac{\mathcal{L}'}{d\mathcal{L}}, \tag{A.8}$$

from which we conclude that $\mathcal{F}_1' > 0$. Then, using $\mathcal{F}_1' > 0$, $\mathcal{F}_1 < 0$, $\mathcal{F}_2' < 0$, and $\mathcal{F}_2 > 0$, we get that the function $\mathcal{F}$ is strictly increasing ($\mathcal{F}' > 0$).

[1] The upper bound of employment $\bar{h}$ coincides with the frictionless employment level $h^\star$ in the case where the marginal banking cost Γ_ℓ converges to zero as real reserves tend to infinity. In general, however, we allow the marginal banking cost to converge to a positive value – in which case we have $\bar{h} < h^\star$, and our economy with the financial friction cannot attain the employment level of the frictionless economy.

Moreover, $\mathcal{F}_1' > 0$ and $\mathcal{F}_1 < 0$ imply that $\lim_{h\to 0} \mathcal{F}_1(h) < 0$, while the Inada condition $\lim_{c\to 0} u'(c) = +\infty$ implies that $\lim_{h\to 0} \mathcal{F}_2(h) = +\infty$, so that

$$\lim_{h\to 0} \mathcal{F}(h) = -\infty.$$

Finally, both $\lim_{h\to\bar{h}} \mathcal{F}_1(h)$ and $\lim_{h\to\bar{h}} \mathcal{F}_2(h)$ are finite, since $\mathcal{F}_1$ is increasing and negative, and $\mathcal{F}_2$ decreasing and positive. If $\bar{h} < h^\star$, then (A.7) and the Inada condition $\lim_{m\to+\infty} \Gamma_m(\ell, m) = 0$ implies $\lim_{h\to\bar{h}} \mathcal{F}_1(h) = 0$. Alternatively, if $\bar{h} = h^\star$, then $\lim_{h\to\bar{h}} \mathcal{F}_2(h) = 0$. We conclude that, in both cases,

$$\lim_{h\to\bar{h}} \mathcal{F}(h) = 0.$$

A.6 Calibration Under Iso-Elastic Production and Utility Functions

To see how our targets for I^ℓ and m/ℓ pin down the parameters ς and V_b, we first rewrite households' first-order conditions (6) and (7) as

$$\beta I^\ell = 1 + \frac{V_b}{(1-\varsigma)\lambda} A_b^{\frac{-(1+\eta)}{1-\varsigma}} \ell^{\frac{\eta+\varsigma}{1-\varsigma}} m^{\frac{-\varsigma(1+\eta)}{1-\varsigma}} \tag{A.9}$$

and

$$\beta I^m = 1 - \frac{\varsigma V_b}{(1-\varsigma)\lambda} A_b^{\frac{-(1+\eta)}{1-\varsigma}} \ell^{\frac{1+\eta}{1-\varsigma}} m^{\frac{-(1+\varsigma\eta)}{1-\varsigma}} \tag{A.10}$$

in the steady state. Equations (A.9) and (A.10) give parameter ς as a function of I^ℓ, m/ℓ, and already calibrated parameters:

$$\varsigma = \left(\frac{m}{\ell}\right)\left(\frac{1-\beta I^m}{\beta I^\ell - 1}\right).$$

Thus, the targets for I^ℓ and m/ℓ pin down ς; we get $\varsigma = 0.0039$.

Next, we rewrite firms' first-order condition under flexible prices (13) as

$$w = \alpha A \left(\frac{\varepsilon - 1}{\varepsilon}\right)\left[\phi\frac{I^\ell}{I} + (1-\phi)\right]^{-1} h^{-(1-\alpha)} \tag{A.11}$$

in the steady state, and we use (3), (9), and (17) to rewrite households' intra-temporal first-order condition (5) as

$$w = V A^\sigma h^{\eta+\alpha\sigma} \tag{A.12}$$

in the steady state. Equations (A.11) and (A.12) give the steady-state value of hours worked h as a function of I^ℓ and already calibrated parameters:

$$h = \left\{\frac{\alpha A^{1-\sigma}}{V}\left(\frac{\varepsilon-1}{\varepsilon}\right)\left[\phi\frac{I^\ell}{I} + (1-\phi)\right]^{-1}\right\}^{\frac{1}{\eta+\alpha\sigma+(1-\alpha)}}.$$

Thus, the target for I^ℓ pins down h. We plug the value obtained for h into either (A.11) or (A.12) to get the steady-state real wage w. Using the borrowing constraint (10) holding with equality, we then get the steady-state value of real loans $\ell = \phi w h$, from which we get in turn the steady-state value of real reserves $m = (m/\ell)\ell$. The value that we have obtained for h also gives us the steady-state value of consumption $c = y = Ah^{\alpha}$, from which we get in turn the steady-state value of the marginal utility of consumption $\lambda = c^{-\sigma}$. By plugging these values of ℓ, m, and λ, as well as the value that we have obtained for ς, into either (A.9) or (A.10), we recover the implied value of V_b; we get $V_b = 0.019$.

Appendix B: Log-Linearized Benchmark Model

In this appendix, we log-linearize our benchmark model around its unique steady state; we show that the model has a unique local equilibrium under exogenous monetary-policy instruments; we characterize this equilibrium; and we derive the necessary and sufficient condition for local-equilibrium determinacy under a floor system.

B.1 Log-Linearization

The IS equation (21), with $\sigma \equiv -u''(c)c/u'(c)$ (where c denotes steady-state consumption), is straightforwardly obtained by log-linearizing households' first-order conditions (3)-(4) and the goods-market-clearing condition (17).

To derive the Phillips curve (22), we log-linearize firms' first-order condition (12), and use the definition of the real wage $w_t \equiv W_t/P_t$, to get

$$\hat{P}_t^* = (1-\beta\theta)\,\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}(\beta\theta)^k\left[\alpha_\phi\left(i_{t+k}^\ell - i_{t+k}\right) + \hat{w}_{t+k} + \hat{P}_{t+k} - \hat{mp}_{t+k|t}\right]\right\}, \tag{B.1}$$

where $\alpha_\phi \equiv \phi I^\ell/[\phi I^\ell + (1-\phi)I] \in (0,1]$, variables with hats denote log deviations from steady-state values, $i_t^\ell \equiv \hat{I}_t^\ell$, $i_t \equiv \hat{I}_t$, and $mp_{t+k|t}$ denotes the marginal productivity in period $t+k$ for a firm whose price was last set in period t. Log-linearizing the production function (9) gives

$$\hat{h}_t = \frac{f}{f'h}\hat{y}_t, \tag{B.2}$$

so that we can rewrite $\hat{mp}_{t+k|t}$ as

$$\begin{aligned}\hat{mp}_{t+k|t} &= \frac{f''h}{f'}\hat{h}_{t+k|t} = \hat{mp}_{t+k} + \frac{f''h}{f'}\left(\hat{h}_{t+k|t} - \hat{h}_{t+k}\right)\\ &= \hat{mp}_{t+k} + \frac{ff''}{(f')^2}\left(\hat{y}_{t+k|t} - \hat{y}_{t+k}\right) = \hat{mp}_{t+k} - \frac{\varepsilon ff''}{(f')^2}\left(\hat{P}_t^* - \hat{P}_{t+k}\right),\end{aligned} \tag{B.3}$$

where mp_{t+k} denotes the average marginal productivity in period $t+k$. Using this result and

$$\pi_t \equiv \log(\Pi_t) = (1-\theta)\left(\hat{P}_t^* - \hat{P}_{t-1}\right),$$

and following the same steps as in, e.g., Galí (2015, Chapter 3), we can rewrite (B.1) as

$$\pi_t = \beta\mathbb{E}_t\{\pi_{t+1}\} + \frac{(1-\theta)(1-\beta\theta)}{\theta\left[1 - \frac{\varepsilon ff''}{(f')^2}\right]}\left[\alpha_\phi\left(i_t^\ell - i_t\right) + \hat{w}_t - \hat{mp}_t\right]. \tag{B.4}$$

Now, log-linearizing the goods-market-clearing condition (17) gives

$$\hat{c}_t = \hat{y}_t. \tag{B.5}$$

Log-linearizing the first-order condition (6), and using (B.5), gives

$$i_t^\ell - i_t = \alpha_\ell \frac{\Gamma_{\ell\ell}\ell}{\Gamma_\ell}\hat{\ell}_t + \alpha_\ell \frac{\Gamma_{\ell m} m}{\Gamma_\ell}\hat{m}_t + \alpha_\ell \sigma \hat{y}_t, \tag{B.6}$$

where $\alpha_\ell \equiv (I^\ell - I)/I^\ell \in (0,1)$. Log-linearizing the first-order condition (5), and using (B.2) and (B.5), gives

$$\hat{w}_t = \left(\sigma + \frac{v''h}{v'}\frac{f}{f'h}\right)\hat{y}_t. \tag{B.7}$$

Log-linearizing the constraint (10) holding with equality, and using (B.2) and (B.7), gives

$$\hat{\ell}_t = \left(\sigma + \frac{v''h}{v'}\frac{f}{f'h} + \frac{f}{f'h}\right)\hat{y}_t. \tag{B.8}$$

Moreover, we have

$$\hat{mp}_t = \frac{ff''}{(f')^2}\left(\hat{y}_t\right). \tag{B.9}$$

Using (B.6), (B.7), (B.8), and (B.9), we can then rewrite (B.4) as the Phillips curve (22):

$$\pi_t = \beta \mathbb{E}_t\left\{\pi_{t+1}\right\} + \kappa\left(\hat{y}_t - \delta_m \hat{m}_t\right)$$

with

$$\begin{aligned}
\kappa &\equiv \left[\sigma + \frac{v''h}{v'}\frac{f}{f'h} - \frac{ff''}{(f')^2} + \alpha_\ell\alpha_\phi\sigma + \alpha_\ell\alpha_\phi\frac{\Gamma_{\ell\ell}\ell}{\Gamma_\ell}\left(\sigma + \frac{v''h}{v'}\frac{f}{f'h} + \frac{f}{f'h}\right)\right]\Psi > 0,\\
\delta_m &\equiv -\alpha_\ell\alpha_\phi\left(\frac{\Gamma_{\ell m} m}{\Gamma_\ell}\right)\frac{\Psi}{\kappa} > 0,
\end{aligned}$$

where in turn

$$\Psi \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta\left[1 - \frac{\varepsilon ff''}{(f')^2}\right]}.$$

To derive the reserves-demand equation (23), we log-linearize the first-order condition (7) and use (B.5) to get

$$i_t - i_t^m = \alpha_m \frac{\Gamma_{\ell m}\ell}{\Gamma_m}\hat{\ell}_t + \alpha_m \frac{\Gamma_{mm} m}{\Gamma_m}\hat{m}_t + \alpha_m \sigma \hat{y}_t, \tag{B.10}$$

where $i_t^m \equiv \hat{I}_t^m$ and $\alpha_m \equiv (I - I^m)/I^m > 0$. Using (B.8), we can rewrite (B.10) as the reserves-demand equation (23):

$$\hat{m}_t = \chi_y \hat{y}_t - \chi_i\left(i_t - i_t^m\right)$$

with

$$\begin{aligned}
\chi_y &\equiv -\left[\sigma + \frac{\Gamma_{\ell m}\ell}{\Gamma_m}\left(\sigma + \frac{v''h}{v'}\frac{f}{f'h} + \frac{f}{f'h}\right)\right]\left(\frac{\Gamma_{mm} m}{\Gamma_m}\right)^{-1} > 0,\\
\chi_i &\equiv \left(-\alpha_m \frac{\Gamma_{mm} m}{\Gamma_m}\right)^{-1} > 0.
\end{aligned}$$

B.2 Unique Local Equilibrium Under Exogenous Instruments

Under permanently exogenous monetary-policy instruments i_t^m and $\hat{M}_t$, the IS equation (21), the Phillips curve (22), the reserves-demand equation (23), and the identities $\hat{m}_t = \hat{M}_t - \hat{P}_t$ and $\pi_t = \hat{P}_t - \hat{P}_{t-1}$ lead to the following dynamic equation relating $\hat{P}_t$ to $\mathbb{E}_t\{\hat{P}_{t+2}\}$, $\mathbb{E}_t\{\hat{P}_{t+1}\}$, $\hat{P}_{t-1}$, and exogenous terms:

$$\mathbb{E}_t\left\{L\mathcal{P}\left(L^{-1}\right)\hat{P}_t\right\} = Z_t,$$

where

$$\begin{aligned}\mathcal{P}(z) \quad &\equiv \quad z^3 - \left[2 + \frac{1}{\beta} + \frac{\chi_y}{\sigma\chi_i} + \left(\frac{1}{\sigma} - \delta_m\right)\frac{\kappa}{\beta}\right]z^2 + \left[1 + \frac{2}{\beta} + \left(1 + \frac{1}{\beta}\right)\frac{\chi_y}{\sigma\chi_i}\right.\\ &\qquad \left. + \left(\frac{1}{\sigma} - \delta_m\right)\frac{\kappa}{\beta} + (1 - \delta_m\chi_y)\frac{\kappa}{\beta\sigma\chi_i}\right]z - \left(\frac{1}{\beta} + \frac{\chi_y}{\beta\sigma\chi_i}\right),\\ Z_t &\equiv \frac{-\kappa}{\beta\sigma}\left(i_t^m - r_t\right) + \left[\frac{1}{\sigma\chi_i} - \left(1 + \frac{\chi_y}{\sigma\chi_i}\right)\delta_m\right]\frac{\kappa}{\beta}\hat{M}_t + \frac{\delta_m\kappa}{\beta}\mathbb{E}_t\left\{\hat{M}_{t+1}\right\}.\end{aligned}$$

Now, our model, given its structure, implies that

$$\sigma < \chi_y < \frac{1}{\delta_m}, \tag{B.11}$$

as we show Appendix B.3. The first inequality in (B.11) arises from the fact that bank loans serve to finance the wage bill (or some fraction of it). If output $\hat{y}_t$ increases by 1%, the marginal utility of consumption decreases by σ%; so, the wage, the wage bill, and loans all increase by more than σ%; and, in turn, so does the demand for reserves $\hat{m}_t$ for a given spread $i_t - i_t^m$ (i.e., $\chi_y > \sigma$). The second inequality in (B.11) reflects how holding reserves mitigates the costs of banking. For a given spread $i_t - i_t^m$, a rise in output $\hat{y}_t$ has two opposite effects on firms' marginal cost of production (i.e., on the term in factor of κ in the Phillips curve): a standard positive direct effect (with elasticity 1), and a negative indirect effect via the implied rise in reserves $\hat{m}_t$ (with elasticity $\chi_y\delta_m$). The inequality states that the direct effect dominates the indirect one (i.e., $\chi_y\delta_m < 1$).

The polynomial $\mathcal{P}(z)$ can be rewritten as

$$\begin{aligned}\mathcal{P}(z) \quad &= \quad z^3 - \left(\frac{1 + 2\beta + \beta\Theta_1 + \Theta_2}{\beta}\right)z^2 + \left[\frac{2 + \beta + (1+\beta)\Theta_1 + \Theta_2 + \Theta_3}{\beta}\right]z - \left(\frac{1 + \Theta_1}{\beta}\right)\\ &= \quad (z - 1 - \Theta_1)\left[z^2 - \left(\frac{1 + \beta + \Theta_2}{\beta}\right)z + \frac{1}{\beta}\right] - \left(\frac{\Theta_1\Theta_2 - \Theta_3}{\beta}\right)z,\end{aligned}$$

where $\Theta_1 \equiv \chi_y/(\sigma\chi_i) > 0$, $\Theta_2 \equiv (1/\sigma - \delta_m)\kappa$, and $\Theta_3 \equiv (1 - \delta_m\chi_y)\kappa/(\sigma\chi_i)$. The double inequality (B.11) implies $\Theta_2 > 0$, $\Theta_3 > 0$, and $\Theta_1\Theta_2 - \Theta_3 = (\chi_y - \sigma)\kappa/(\sigma^2\chi_i) > 0$. Therefore, we get $\mathcal{P}(0) = -(1+\Theta_1)/\beta < 0$, $\mathcal{P}(1) = \Theta_3/\beta > 0$, $\mathcal{P}(1+\Theta_1) = -(\Theta_1\Theta_2 - \Theta_3)(1+\Theta_1)/\beta < 0$, and $\lim_{z\in\mathbb{R}, z\to+\infty}\mathcal{P}(z) = +\infty > 0$. As a consequence, the roots of $\mathcal{P}(z)$ are three real numbers ρ, ω_1, and ω_2 such that $0 < \rho < 1 < \omega_1 < 1 + \Theta_1 < \omega_2$.

With one eigenvalue inside the unit circle (ρ) for one predetermined variable ($\hat{P}_{t-1}$), thus, our model satisfies Blanchard and Kahn's (1980) conditions and has a unique bounded solution. We rewrite the dynamic equation as

$$\mathbb{E}_t \left\{ \left(L^{-1} - \omega_1\right) \left(L^{-1} - \omega_2\right) (1 - \rho L) \hat{P}_t \right\} = Z_t$$

and use the method of partial fractions to solve this equation forward and get the unique bounded solution for $\hat{P}_t - \rho \hat{P}_{t-1}$:

$$\begin{aligned} \hat{P}_t - \rho \hat{P}_{t-1} &= \mathbb{E}_t \left\{ \frac{Z_t}{(L^{-1} - \omega_1)(L^{-1} - \omega_2)} \right\} = \frac{\mathbb{E}_t}{\omega_2 - \omega_1} \left\{ \frac{\omega_1^{-1} Z_t}{1 - (\omega_1 L)^{-1}} - \frac{\omega_2^{-1} Z_t}{1 - (\omega_2 L)^{-1}} \right\} \\ &= \frac{\mathbb{E}_t}{\omega_2 - \omega_1} \left\{ \sum_{k=0}^{+\infty} \left(\omega_1^{-k-1} - \omega_2^{-k-1} \right) Z_{t+k} \right\}. \end{aligned} \tag{B.12}$$

Using the price-level solution (B.12), the Phillips curve (22), and the identities $\hat{m}_t = \hat{M}_t - \hat{P}_t$ and $\pi_t = \hat{P}_t - \hat{P}_{t-1}$, we then get

$$\begin{aligned} \pi_t &= -(1-\rho)\hat{P}_{t-1} + \frac{\mathbb{E}_t}{\omega_2 - \omega_1} \left\{ \sum_{k=0}^{+\infty} \left(\omega_1^{-k-1} - \omega_2^{-k-1} \right) Z_{t+k} \right\}, \\ \hat{y}_t &= -\vartheta \hat{P}_{t-1} + \delta_m \hat{M}_t - \frac{\mathbb{E}_t}{(\omega_2 - \omega_1)\kappa} \left\{ \sum_{k=0}^{+\infty} \left(\xi_1 \omega_1^{-k-1} - \xi_2 \omega_2^{-k-1} \right) Z_{t+k} \right\}, \end{aligned}$$

where $\vartheta \equiv (1-\rho)(1-\beta\rho)/\kappa + \delta_m \rho$ and $\xi_j \equiv \beta(\omega_j + \rho - 1) + \kappa\delta_m - 1$ for $j \in \{1, 2\}$.

B.3 Proof of the Double Inequality (B.11)

To show that $\chi_y < 1/\delta_m$, we define $\Omega \equiv \delta_m \kappa / (\alpha_m \chi_i \Psi) > 0$ and we write

$$\begin{aligned} \Omega \left(\frac{1}{\delta_m} - \chi_y \right) &= \frac{-\Gamma_{mm} m}{\Gamma_m} \left[\alpha_\ell \alpha_\phi \frac{\Gamma_{\ell\ell} \ell}{\Gamma_\ell} \left(\sigma + \frac{v'' h}{v'} \frac{f}{f' h} + \frac{f}{f' h} \right) \right. \\ &\quad \left. + (1 + \alpha_\ell \alpha_\phi) \sigma + \frac{v'' h}{v'} \frac{f}{f' h} - \frac{f f''}{(f')^2} \right] \\ &\quad + \alpha_\ell \alpha_\phi \frac{\Gamma_{\ell m} m}{\Gamma_\ell} \left[\sigma + \frac{\Gamma_{\ell m} \ell}{\Gamma_m} \left(\sigma + \frac{v'' h}{v'} \frac{f}{f' h} + \frac{f}{f' h} \right) \right] \\ &= \frac{-\Gamma_{mm} m}{\Gamma_m} \left[\sigma + \frac{v'' h}{v'} \frac{f}{f' h} - \frac{f f''}{(f')^2} \right] \\ &\quad - \frac{\alpha_\ell \alpha_\phi \ell m}{\Gamma_\ell \Gamma_m} \left(\sigma + \frac{v'' h}{v'} \frac{f}{f' h} + \frac{f}{f' h} \right) \left[\Gamma_{\ell\ell} \Gamma_{mm} - (\Gamma_{\ell m})^2 \right] \\ &\quad + \alpha_\ell \alpha_\phi \sigma m \left(\frac{\Gamma_{\ell m}}{\Gamma_\ell} - \frac{\Gamma_{mm}}{\Gamma_m} \right). \end{aligned}$$

The last expression is the sum of three terms (one per line). The first term is positive. So is the second one, given (A.5). And so is the third one, given that

$$\frac{\Gamma_{\ell m}}{\Gamma_\ell} - \frac{\Gamma_{mm}}{\Gamma_m} = \frac{\left(v^{b\prime}\right)^2 \left(g_m^b g_{\ell m}^b - g_\ell^b g_{mm}^b \right)}{\Gamma_\ell \Gamma_m} = \frac{\left(v^{b\prime}\right)^2 \left(f_h^b f_{mm}^b - f_m^b f_{hm}^b \right)}{\Gamma_\ell \Gamma_m \left(f_h^b\right)^3} > 0. \tag{B.13}$$

Therefore, the whole expression is positive, which implies that $\chi_y < 1/\delta_m$.

To show that $\sigma < \chi_y$, we write

$$\begin{aligned}\frac{1}{\alpha_m \sigma \chi_i}(\chi_y - \sigma) &= \frac{\Gamma_{mm} m}{\Gamma_m} + 1 + \frac{1}{\sigma}\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\left(\sigma + \frac{v'' h}{v'}\frac{f}{f' h} + \frac{f}{f' h}\right) \\ &= 1 + \frac{1}{\sigma}\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\left(\frac{v'' h}{v'}\frac{f}{f' h} + \frac{f}{f' h}\right) + \frac{1}{\Gamma_m}\left(\ell \Gamma_{\ell m} + m \Gamma_{mm}\right).\end{aligned}$$

The last expression is the sum of three terms. The first two terms are positive. And so is the third one, given that

$$\begin{aligned}\ell \Gamma_{\ell m} + m \Gamma_{mm} &= (1-d)\,\ell \Gamma_{\ell m} + d \ell \Gamma_{\ell m} + m \Gamma_{mm} \\ &= (1-d)\,\ell \Gamma_{\ell m} + d\ell\left[v^{b\prime\prime} g_\ell^b g_m^b + v^{b\prime} g_{\ell m}^b\right] + m\left[v^{b\prime\prime}\left(g_m^b\right)^2 + v^{b\prime} g_{mm}^b\right] \\ &= (1-d)\,\ell \Gamma_{\ell m} + v^{b\prime\prime} g_m^b\left(d\ell g_\ell^b + m g_m^b\right) + v^{b\prime}\left(d\ell g_{\ell m}^b + m g_{mm}^b\right) \\ &= (1-d)\,\ell \Gamma_{\ell m} + v^{b\prime\prime} g^b g_m^b \\ &\leq 0, \end{aligned} \tag{B.14}$$

where the last equality follows from (A.2) and (A.4). Therefore, the whole expression is positive, which implies that $\sigma < \chi_y$.

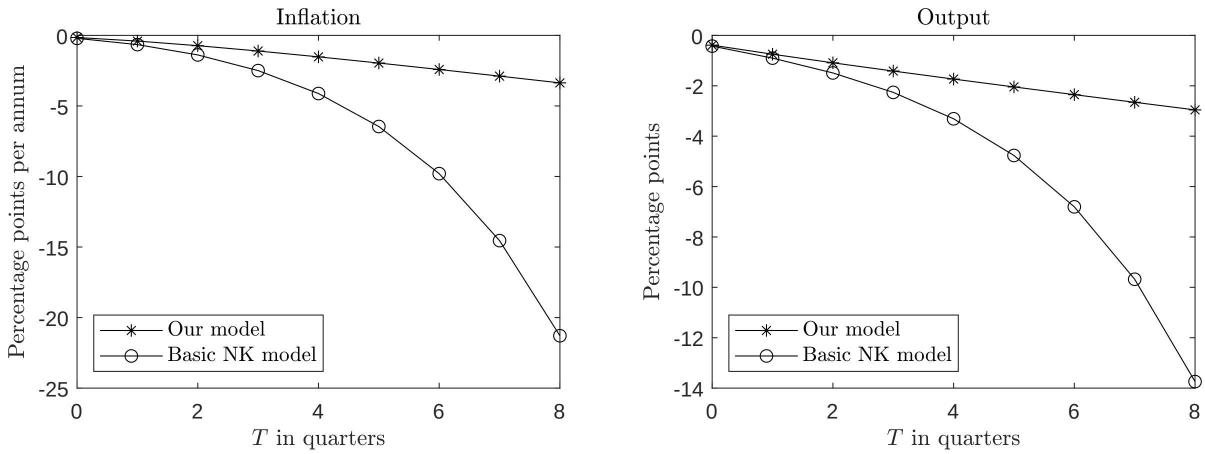
B.4 Comparison With the Basic NK Model

In Subsection 3.3 of the main text, we contrast the implications of our model for inflation during a temporary ZLB episode with the implications of the basic NK model. More specifically, we show that the deflation rate at the start of the ZLB episode grows exponentially with the expected duration of the ZLB episode in the basic NK model, while in our model it converges to a finite value as the expected duration of the ZLB episode goes to infinity.

In this appendix, we illustrate these results numerically. We consider the same calibration of our model as in Subsection 4.1, and we consider the corresponding calibration of the basic NK model, i.e. we set the structural parameters of the basic NK model to the values that they take in our calibrated model (given that all the structural parameters of the basic NK model are also structural parameters of our model). In both models, we set the value of the discount-factor shock (leading to the ZLB) based on Cúrdia's (2015) average estimate of the natural rate of interest over the 2009Q1-2015Q4 ZLB episode. This average estimate is -3.1% per year; average CPI inflation over the same period is 1.4% per year; so, we get an estimate of Cúrdia's "interest-rate gap" at -3.1+1.4=-1.7% per year, and we thus set $i_t^m - r_t$ to 1.7% per year during our ZLB episode (i.e. $z^* = 1.7\%$ in the notation of Subsection 3.3).

Figure B.1 shows the value of inflation and output at the start of the ZLB episode, depending on the expected duration T of the ZLB episode, in the basic NK model and in our model. In the basic NK model (but not in our model), inflation and output decrease exponentially with

Figure B.1 – Effects of a ZLB episode of expected duration T on inflation and output at the start of the episode



T, and their fall quickly reaches substantial values: -21% per year for inflation and -14% for output for an expected ZLB duration of 8 quarters (i.e. 2 years).

Eight quarters for the expected ZLB duration does not seem an unrealistic figure to us. The Survey of Primary Dealers of the Federal Reserve Bank of New York indicates that in September 2012, primary dealers were expecting the ZLB episode to last for the next 11 quarters (until mid-2015), as emphasized by Yellen (2012). In October and December 2012, as well as in January 2013, they were still expecting the ZLB episode to last for the next 10 quarters.[2]

B.5 Determinacy Condition Under a Floor System

Using the IS equation (21), the Phillips curve (22), the reserves-demand equation (23), the Taylor rule (26), and the identities $\hat{m}_t = \hat{M}_t - \hat{P}_t$ and $\pi_t = \hat{P}_t - \hat{P}_{t-1}$, we get a dynamic equation relating $\hat{P}_t$ to $\mathbb{E}_t\{\hat{P}_{t+2}\}$, $\mathbb{E}_t\{\hat{P}_{t+1}\}$, $\hat{P}_{t-1}$, and exogenous terms, whose characteristic polynomial is

$$\mathcal{P}_r(z) \equiv z^3 - a_2 z^2 + a_1 z - a_0$$

with

$$\begin{aligned} a_2 &\equiv 2 + \frac{1}{\beta} + \frac{(1-\sigma\delta_m)\,\kappa}{\beta\sigma} + \frac{\chi_y}{\sigma\chi_i} + \frac{r_y}{\sigma} > 0, \\ a_1 &\equiv 1 + \frac{2}{\beta} + \frac{(1-\sigma\delta_m)\,\kappa}{\beta\sigma} + \frac{(1+\beta)\,\chi_y}{\beta\sigma\chi_i} + \frac{(1-\delta_m\chi_y)\,\kappa}{\beta\sigma\chi_i} + \frac{\kappa r_\pi}{\beta\sigma} + \frac{(1+\beta-\delta_m\kappa)\,r_y}{\beta\sigma}, \\ a_0 &\equiv \frac{1}{\beta} + \frac{\chi_y}{\beta\sigma\chi_i} + \frac{\kappa r_\pi}{\beta\sigma} + \frac{r_y}{\beta\sigma} > 0, \end{aligned}$$

where the first inequality follows from the double inequality (B.11). Given that there is exactly one predetermined variable ($\hat{P}_{t-1}$), the necessary and sufficient condition for local-equilibrium

[2] Similarly, the Blue-Chip Survey of Forecasters indicates that in several months of 2011-2013, the Fed Funds rate was expected to remain constant for "7 or more" quarters. And the Philadelphia Fed's Survey of Professional Forecasters indicates that in 2012Q1, the 3-month T-bill rate was expected to remain at 0.10% for at least 7 quarters (i.e. at least throughout both 2012 and 2013).

determinacy is that $\mathcal{P}_r(z)$ have exactly one root inside the unit circle. This root must be a real number (indeed, if it were a complex number, its conjugate would be another root inside the unit circle). We have $\mathcal{P}_r(0) = -a_0 < 0$ and $\mathcal{P}_r(1) = (1 - \delta_m\chi_y - \delta_m\chi_i r_y)\kappa/(\beta\sigma\chi_i)$. In the following, we consider two alternative cases in turn, depending on the sign of $\mathcal{P}_r(1)$.

We first consider the case in which $\mathcal{P}_r(1) > 0$, that is to say equivalently the case in which

$$r_y < \zeta_1, \tag{B.15}$$

where

$$\zeta_1 \equiv \frac{1 - \delta_m\chi_y}{\delta_m\chi_i} > 0,$$

where in turn the last inequality follows from the second inequality in (B.11). In this case, $\mathcal{P}_r(0)$ and $\mathcal{P}_r(1)$ are of opposite signs, so that $\mathcal{P}_r(z)$ has either one or three real roots inside $(0, 1)$. Moreover, in this case, we have

$$a_1 = 1 + \frac{2}{\beta} + \frac{(1 - \sigma\delta_m)\,\kappa}{\beta\sigma} + \frac{(1 + \beta)\,\chi_y}{\beta\sigma\chi_i} + \frac{\kappa r_\pi}{\beta\sigma} + \frac{(1 + \beta)\,r_y}{\beta\sigma} + \frac{(\zeta_1 - r_y)\,\delta_m\kappa}{\beta\sigma} > 0,$$

where the inequality comes from (B.11) and (B.15). In turn, $a_1 > 0$, together with $a_0 > 0$ and $a_2 > 0$, implies that $\mathcal{P}_r(z) < 0$ for all $z < 0$, and hence that $\mathcal{P}_r(z)$ has no negative real roots. So, $\mathcal{P}_r(z)$ has at least one real root inside $(0, 1)$, which we denote by ρ, and its other two roots, which we denote by ω_1 and ω_2 with $|\omega_1| \leq |\omega_2|$, are either (i) both real and inside $(0, 1)$, or (ii) both real and higher than 1, or (iii) both complex and conjugates of each other. Now, we have $\rho + \omega_1 + \omega_2 = a_2 > 3$, where the inequality follows from the double inequality (B.11) and from $\beta < 1$. Therefore, Case (i) is impossible, and in Case (iii) the common real part of ω_1 and ω_2 is higher than 1. So, in the remaining two possible cases, namely Cases (ii) and (iii), ω_1 and ω_2 lie outside the unit circle. As a consequence, we get local-equilibrium determinacy.

We now turn to the alternative case in which $\mathcal{P}_r(1) < 0$, that is to say equivalently the case in which Condition (B.15) is not met. In this case, $\mathcal{P}_r(0)$ and $\mathcal{P}_r(1)$ have the same sign, so that $\mathcal{P}_r(z)$ has either zero or two real roots inside $(0, 1)$. Therefore, a necessary condition for local-equilibrium determinacy is then that $\mathcal{P}_r(-1)$ be of the opposite sign, i.e. $\mathcal{P}_r(-1) > 0$, so that $\mathcal{P}_r(z)$ can have either one or three real roots inside $(-1, 0)$. This necessary condition for determinacy can be written as

$$[\delta_m\kappa - 2\,(1 + \beta)]\,r_y > 4\,(1 + \beta)\,\sigma + \frac{2\,(1 + \beta)\,\chi_y}{\chi_i} + 2\,(1 - \sigma\delta_m)\,\kappa + \frac{(1 - \delta_m\chi_y)\,\kappa}{\chi_i} + 2\kappa r_\pi.$$

The right-hand side of this inequality is positive, given the double inequality (B.11). Therefore, the necessary condition for determinacy can be equivalently rewritten as

$$\delta_m\kappa > 2\,(1 + \beta) \quad \text{and} \quad r_y > \zeta_2 + \zeta_3 r_\pi, \tag{B.16}$$

where

$$\zeta_2 \equiv \frac{4\,(1 + \beta)\,\sigma\chi_i + 2\,(1 + \beta)\,\chi_y + 2\,(1 - \sigma\delta_m)\,\kappa\chi_i + (1 - \delta_m\chi_y)\,\kappa}{[\delta_m\kappa - 2\,(1 + \beta)]\,\chi_i} > \zeta_1,$$

$$\zeta_3 \equiv \frac{2\kappa}{\delta_m\kappa - 2\,(1 + \beta)} > 0,$$

where in turn the last two inequalities follow from the first inequality in (B.16). We now show that Condition (B.16) is not only necessary, but also sufficient for local-equilibrium determinacy in that case. To that aim, assume that this condition is met. Then, $\mathcal{P}_r(-1)$ and $\mathcal{P}_r(0)$ are of opposite signs, so that $\mathcal{P}_r(z)$ has either one or three real roots inside $(-1, 0)$. Let ρ denote one root of $\mathcal{P}_r(z)$ inside $(-1, 0)$. The other two roots of $\mathcal{P}_r(z)$, which we denote by ω_1 and ω_2 with $|\omega_1| \leq |\omega_2|$, can be either (i) both real and inside $(-1, 0)$, or (ii) both real and inside $(0, 1)$, or (iii) both real and outside $(-1, 1)$, or (iv) both complex and conjugates of each other. Since $\rho + \omega_1 + \omega_2 = a_2 > 3$, however, Cases (i) and (ii) are impossible, and in Case (iv) the common real part of ω_1 and ω_2 is higher than 1. Therefore, in the remaining two possible cases, namely Cases (iii) and (iv), ω_1 and ω_2 lie outside the unit circle. As a consequence, Condition (B.16) is, indeed, sufficient for local-equilibrium determinacy in that case.

From the results obtained in the two alternative cases considered, we get that there is local-equilibrium determinacy if and only if either Condition (B.15) is met, or Condition (B.15) is not met and Condition (B.16) is met. Now, Conditions (B.15) and (B.16) are mutually exclusive, given that $\zeta_2 > \zeta_1$. We conclude that there is local-equilibrium determinacy if and only if either Condition (B.15) or Condition (B.16) is met.

B.6 Floor system and supply shock

In this appendix, motivated in part by the surge in US inflation in 2021-2022, we introduce a supply shock A_t into our model and compare the performance of alternative rules that link the policy rate to inflation. We rewrite the production function as

$$y_t(i) = A_t f[h_t(i)].$$

We log-linearize the model in the same way as in Appendix B.1. The IS equation (21) is unchanged, but a new term in $\hat{A}_t$ appears in the Phillips curve and the reserves-market-clearing condition:

$$\begin{aligned} \pi_t &= \beta \mathbb{E}_t \{\pi_{t+1}\} + \kappa \left(\hat{y}_t - \delta_m \hat{m}_t - \delta_a \hat{A}_t \right), \\ \hat{m}_t &= \chi_y \hat{y}_t - \chi_i (i_t - i_t^m) - \chi_a \hat{A}_t, \end{aligned}$$

where

$$\begin{aligned} \delta_a &\equiv \left[1 + \frac{v''h}{v'} \frac{f}{f'h} - \frac{ff''}{(f')^2} + \alpha_\ell \alpha_\phi \frac{\Gamma_{\ell\ell}\ell}{\Gamma_\ell} \left(\frac{v''h}{v'} \frac{f}{f'h} + \frac{f}{f'h} \right) \right] \frac{\Psi}{\kappa} > 0 \\ \chi_a &\equiv \alpha_m \frac{\Gamma_{\ell m}\ell}{\Gamma_m} \left(\frac{v''h}{v'} \frac{f}{f'h} + \frac{f}{f'h} \right) \chi_i > 0. \end{aligned}$$

A positive supply shock ($\hat{A}_t > 0$) reduces the marginal cost of production for a given output level and a given level of real reserve balances (hence the negative term $-\delta_a \hat{A}_t$ in the Phillips curve), and it also reduces the demand for reserves for a given output level and a given spread (hence the negative term $-\chi_a \hat{A}_t$ in the reserves-market-clearing condition).

We consider a floor system in which M_t is set exogenously and i_t^m is set according to the simple rule

$$i_t^m = r_\pi \pi_t,$$

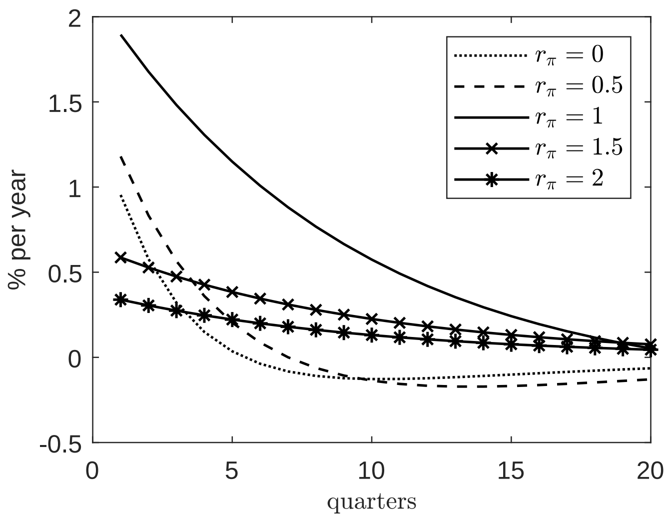
where $r_\pi \geq 0$. This floor system delivers local-equilibrium determinacy, as we have shown in Section 6.2 and Appendix B.5. Finally, we assume that $\hat{A}_t$ follows an AR(1) process:

$$\hat{A}_t = \rho_a \hat{A}_{t-1},$$

where $\rho_a \in [0, 1]$.

For the standard value $\rho_a = 0.9$, Figure B.2 shows the response of inflation to a 1% negative supply shock under alternative policies with different values of r_π. Interestingly, the performance of the policy rule deteriorates – in terms of stabilizing inflation over horizons up to 10 quarters – as we go from $r_\pi = 0$ to $r_\pi = 0.5$, and then to $r_\pi = 1$.

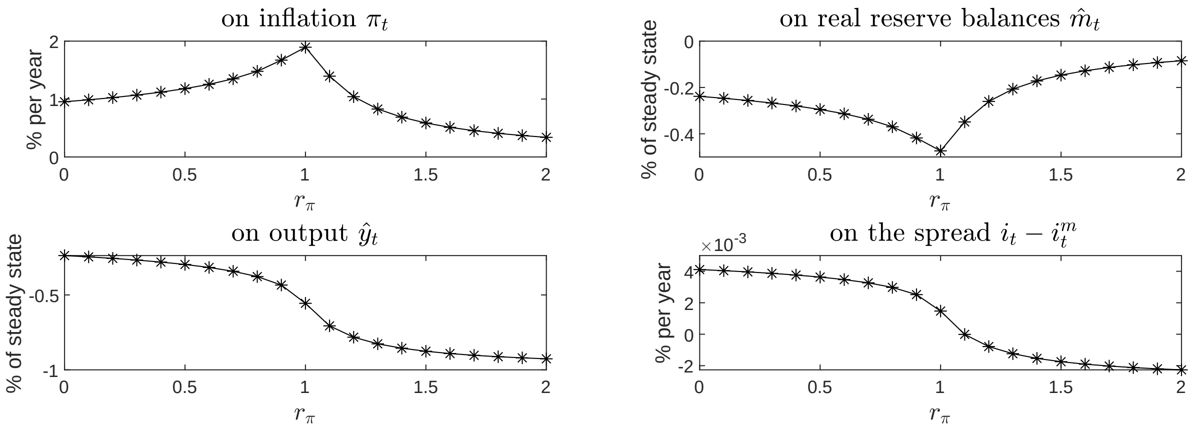
Figure B.2 – Response of inflation to a 1% negative supply shock, depending on r_π



Changing r_π affects inflation through two channels in our model. One channel is the familiar one: for $r_\pi > 1$, an inflationary shock leads to an increase in the *real* policy rate. The second channel is a cost channel of monetary policy reflected in Equations (22) and (23). A stronger response to inflation leads to a sharper drop in output, which leads to a sharper drop in real money balances in (23). The sharper drop in real money balances, in turn, raises banking costs, which raises inflation in (22).

To be more precise about how the cost channel works in our model, we focus on the *impact* effects of the 1% adverse supply shock under alternative rules. Figure B.3 shows the contemporaneous impact of the shock on inflation, real reserve balances, output and the spread, depending on r_π. Since nominal reserve balances are constant, the impact on inflation (top left panel of Figure B.3) is the opposite of the impact on real reserve balances (top right panel of Figure B.3), up to a factor 4 (as inflation is expressed in % per year, not in % per quarter). The inflationary impact of the adverse supply shock is non-monotonic in the rule's coefficient r_π: it first increases with r_π for r_π below one, and then decreases with r_π for r_π above one (top left panel of Figure B.3). This non-monotonicity can be understood as the result of two opposite effects: an output effect, and a spread effect.

Figure B.3 – Impact of a 1% negative supply shock, depending on r_π



The output effect is the following: for a given increase in inflation (triggered by the adverse supply shock), as r_π increases, I_t^m rises further, and so does I_t. The increase in I_t has a contractionary effect on output (bottom left panel of Figure B.3). For a given spread $I_t - I_t^m$, this output contraction makes the demand for real reserve balances fall in the reserves-market-clearing condition (23). Since nominal reserve balances are constant, prices rise. So, the output effect makes inflation increase with r_π, for a given spread.

The spread effect is that the increase in r_π directly raises I_t^m, which in turn indirectly raises I_t, but I_t rises by less than I_t^m; so, the spread $I_t - I_t^m$ is compressed (bottom right panel of Figure B.3). For a given output y_t, this spread compression makes the demand for real reserve balances increase in the reserves-market-clearing condition (23). Since nominal reserve balances are constant, prices fall. So, the spread effect makes inflation decrease with r_π, for a given output.

The response of inflation to the adverse supply shock can thus increase or decrease with the rule's coefficient r_π, depending on which of the two (output and spread) effects dominates the other. To further understand why inflation increases (resp. decreases) with r_π for small (resp. large) values of r_π, we can solve for the unique local equilibrium of the model in the same way as in Appendix B.2; we then get

$$\begin{aligned}
\pi_t &= -(1-\rho)\hat{P}_{t-1} + \frac{(1-\delta_m\chi_y)\kappa}{\beta\sigma\chi_i(\omega_1-1)(\omega_2-1)}\hat{M}_{t-1} \\
&+\frac{\kappa}{\beta(\omega_2-\omega_1)}\mathbb{E}_t\left\{\frac{1}{\sigma}\sum_{k=0}^{+\infty}\left(\omega_1^{-k-1}-\omega_2^{-k-1}\right)r_{t+k}\right. \\
&+\sum_{k=0}^{+\infty}\left[\left(\frac{1-\delta_m\chi_y}{\sigma\chi_i}\right)\left(\frac{\omega_1^{-k}}{\omega_1-1}-\frac{\omega_2^{-k}}{\omega_2-1}\right)+\delta_m\left(\omega_1^{-k}-\omega_2^{-k}\right)\right]\hat{\mu}_{t+k} \\
&\left.+\sum_{k=0}^{+\infty}\left[\left(\frac{\chi_a-\delta_a\chi_y}{\sigma\chi_i}\right)\left(\frac{\omega_1^{-k}}{\omega_1-1}-\frac{\omega_2^{-k}}{\omega_2-1}\right)+\delta_a\left(\omega_1^{-k}-\omega_2^{-k}\right)\right]\Delta\hat{A}_{t+k}\right\}, \quad \text{(B.17)}
\end{aligned}$$

where ρ, ω_1 and ω_2 denote again the three roots of the characteristic polynomial, with $|\rho| < 1 < |\omega_1| \leq |\omega_2|$. There are two changes compared to Equation (25). First, inflation now responds to the (newly introduced) supply shock $\hat{A}_t$. Second, the roots ρ, ω_1 and ω_2, which characterize the dynamic responses of inflation to all shocks, now depend on the coefficient r_π of the IOR-rate rule.

Focusing on the supply shock (i.e. setting $\hat{M}_{t-1} = 0$ and $r_{t+k} = \hat{\mu}_{t+k} = 0$ for all $k \geq 0$) and using the AR(1) assumption for this shock (i.e. using $\mathbb{E}_t\{\Delta\hat{A}_{t+k}\} = -(1-\rho_a)\rho_a^{k-1}\hat{A}_t$ for all $k \geq 1$), we can rewrite (B.17) as

$$\begin{aligned}
\pi_t &= -(1-\rho)\hat{P}_{t-1} + \frac{(\delta_a\chi_y-\chi_a)\kappa}{\beta\sigma\chi_i(\omega_1-1)(\omega_2-1)}\hat{A}_{t-1} \\
&-\frac{\kappa}{\beta(\omega_1-\rho_a)(\omega_2-\rho_a)}\left[\frac{\delta_a\chi_y-\chi_a}{\sigma\chi_i}+\delta_a(1-\rho_a)\right]\hat{A}_t. \quad \text{(B.18)}
\end{aligned}$$

It is easy to show that $\delta_a\chi_y - \chi_a > 0$ for iso-elastic production and utility functions. So, the coefficient of $\hat{A}_t$ in (B.18) is always negative:

$$\frac{\partial\pi_t}{\partial\hat{A}_t} = -\frac{\kappa}{\beta(\omega_1-\rho_a)(\omega_2-\rho_a)}\left[\frac{\delta_a\chi_y-\chi_a}{\sigma\chi_i}+\delta_a(1-\rho_a)\right] < 0,$$

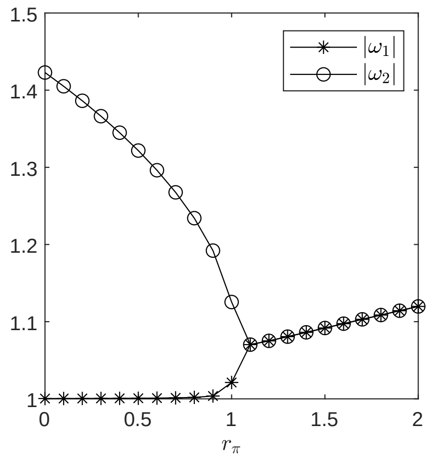
that is to say that a negative supply shock ($\hat{A}_t < 0$) always raises inflation ($\pi_t > 0$).

The only way in which r_π affects $\partial\pi_t/\partial\hat{A}_t$ is via ω_1 and ω_2. We still have $1 < |\omega_1| \leq |\omega_2|$ (as we had for $r_\pi = 0$), but now ω_1 and ω_2 may be complex conjugates. For the calibration that we use in the paper, as r_π changes, $|\omega_1|$ and $|\omega_2|$ move as described in Figure B.4.

For r_π between 0 and 1, ω_1 and ω_2 are real numbers, with $1 < \omega_1 < \omega_2$; ω_1 varies very little (remains very close to 1), while ω_2 decreases with r_π; so, $\partial\pi_t/\partial\hat{A}_t$ increases with r_π. Alternatively, for r_π between 1.1 and 2, ω_1 and ω_2 are complex conjugates, and their common modulus $|\omega_1| = |\omega_2|$ increases with r_π; so, $\partial\pi_t/\partial\hat{A}_t$ decreases with r_π.

The result that we have emphasized – that the inflationary impact of the adverse supply shock is maximal when the rule's coefficient is around one – is robust to alternative specifications for

Figure B.4 – $|\omega_1|$ and $|\omega_2|$, depending on r_π

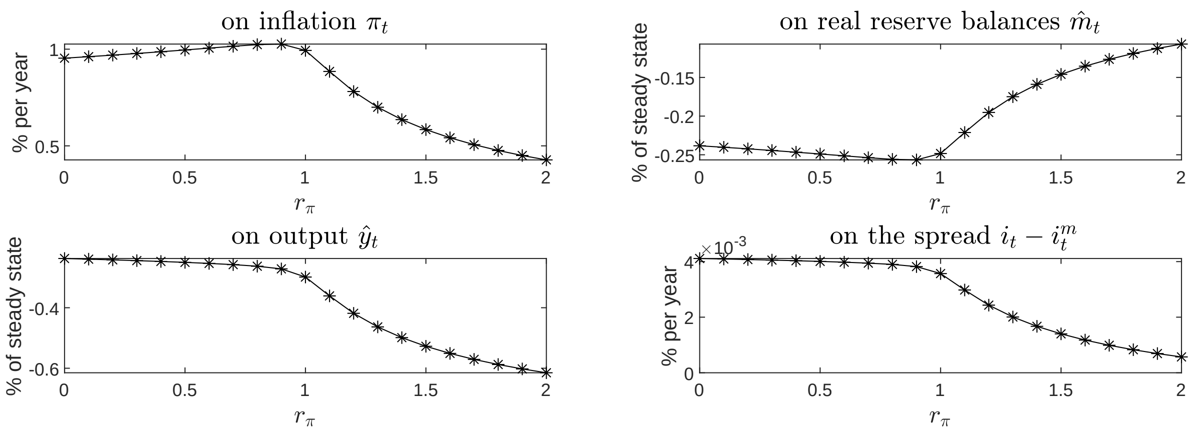


the IOR-rate rule and alternative model calibrations. We consider two other rules:

$$
\begin{aligned}
i_t^m &= r_\pi \pi_{t-4}, \\
i_t^m &= 0.8 i_{t-1}^m + 0.2 r_\pi \pi_t,
\end{aligned}
$$

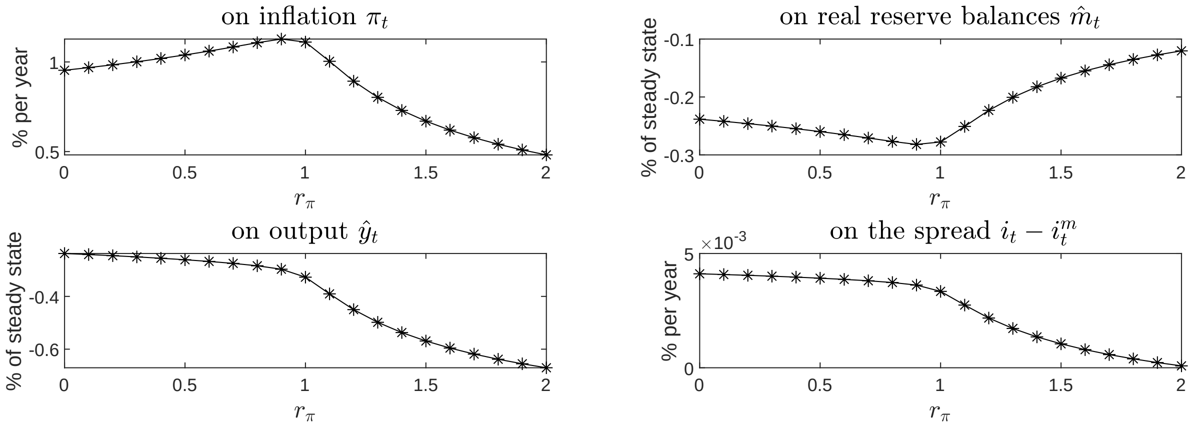
where $r_\pi \geq 0$. The first (lagged) rule is motivated by the fact that central banks started to raise rates in 2022Q2, about four quarters after inflation started to surge (2021Q2). The second (inertial) rule is commonly considered in the literature. Figures B.5 and B.6 are the counterparts of Figure B.3 for these alternative rules. The three figures are broadly similar, although the rise of the inflationary impact of the shock as a function of r_π for $0 < r_\pi < 1$ is less pronounced in Figures B.5 and B.6 than in Figure B.3.

Figure B.5 – Impact of a 1% negative supply shock (lagged rule), depending on r_π



We also consider a range of alternative values for the persistence parameter ρ_a of the supply shock and for the steady-state spread $I - I^m$ (which measures how far we are from satiation at the steady state). Values of ρ_a ranging from 0 to 0.99 (benchmark: 0.9) may change the results

Figure B.6 – Impact of a 1% negative supply shock (inertial rule), depending on r_π



quantitatively, but not qualitatively. Values of $I - I^m$ ranging from 5 to 50 basis points per annum (benchmark: 10 basis points per annum) do not change the results, neither quantitatively nor qualitatively.

Appendix C: Benchmark Model With Reserves-Supply Rule

In the main text (Section 3), we show that our benchmark model delivers local-equilibrium determinacy under exogenous monetary-policy instruments, and we use this determinacy result to explain the low volatility of inflation and the absence of significant deflation at the ZLB. Our (simplifying) assumption of exogenous nominal reserves does not seem to us like a bad approximation of reality, given how the Fed has announced in advance a path for its balance sheet. Nonetheless, the alternative assumption of endogenous nominal reserves (i.e. a reserves-supply rule) seems also relevant. In the present appendix, we show that our determinacy result is essentially robust to the endogenization of nominal reserves. More specifically, we consider a reserves-supply rule in our benchmark model; we derive a sufficient condition for local-equilibrium determinacy under this rule and an exogenous IOR rate; and we argue that this sufficient determinacy condition is likely to be met.

C.1 Reserves-Supply Rule, Steady State, and Log-Linearization

We assume that the central bank sets the stock of nominal reserves according to the rule

$$M_t = P_t \mathcal{Q}\left(P_t, y_t\right), \tag{C.1}$$

where the function $\mathcal{Q}$, from $\mathbb{R}^2_{>0}$ to $\mathbb{R}_{>0}$, is differentiable, decreasing in P_t ($\mathcal{Q}_P < 0$), and non-increasing in y_t ($\mathcal{Q}_y \leq 0$). This assumption ensures that real reserve balances respond negatively to the price level for a given output level, and non-positively to the output level for a given price level. This specification nests, in particular, the case of (constant) exogenous nominal reserves considered in the rest of the paper, which corresponds to $\mathcal{Q}(P_t, y_t) = M/P_t$ with $M > 0$.

This reserves-supply rule does not change any of the steady-state-equilibrium conditions stated in Subsection 3.1. In particular, in any steady state, because real money balances and real output are constant over time (by definition of a steady state), the reserves-supply rule (C.1) implies that the price level is constant over time as well, like previously. Therefore, the necessary and sufficient condition on I^m for existence and uniqueness of a steady state is still $I^m < 1/\beta$.

Log-linearizing the model around its unique steady state, we get the same IS equation (21), Phillips curve (22), and reserves-demand equation (23) as previously, plus now the rule

$$\hat{m}_t = -q_P \hat{P}_t - q_y \hat{y}_t, \tag{C.2}$$

where $q_P \equiv -P\mathcal{Q}_P(P,y)/\mathcal{Q}(P,y) > 0$ and $q_y \equiv -y\mathcal{Q}_y(P,y)/\mathcal{Q}(P,y) \geq 0$ (variables without time subscript denote steady-state values).

C.2 Derivation of a Sufficient Determinacy Condition

Using the four log-linearized equations just mentioned and the identity $\pi_t = \hat{P}_t - \hat{P}_{t-1}$, we get the following dynamic system in $\hat{m}_t$ and $\hat{y}_t$:

$$\mathbb{E}_t \left\{ \begin{bmatrix} \hat{m}_{t+1} \\ \hat{m}_t \\ \hat{y}_{t+1} \\ \hat{y}_t \end{bmatrix} \right\} = \mathbf{A} \begin{bmatrix} \hat{m}_t \\ \hat{m}_{t-1} \\ \hat{y}_t \\ \hat{y}_{t-1} \end{bmatrix} + \mathbf{B} \begin{bmatrix} i_t^m \\ r_t \end{bmatrix}, \tag{C.3}$$

where

$$\begin{aligned} \mathbf{A} \equiv & \begin{bmatrix} \frac{1+\beta-\delta_m\kappa}{\beta} & \frac{-1}{\beta} & \frac{\kappa}{\beta} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1-\delta_m\kappa}{\beta\sigma} - \frac{1}{\sigma\chi_i} & \frac{-1}{\beta\sigma} & 1 + \frac{\chi_y}{\sigma\chi_i} + \frac{\kappa}{\beta\sigma} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + (q_P - 1) \begin{bmatrix} \frac{-\delta_m\kappa}{\beta} & 0 & \frac{\kappa}{\beta} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ & + \frac{q_P - 1}{q_P} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-1}{\beta\sigma} & \frac{1}{\beta\sigma} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + q_y \begin{bmatrix} \frac{\delta_m\kappa}{\beta\sigma} + \frac{1}{\sigma\chi_i} & 0 & \frac{1}{\beta} - \frac{\kappa}{\beta\sigma} - \frac{\chi_y}{\sigma\chi_i} & \frac{-1}{\beta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ & + \frac{q_y}{q_P} \begin{bmatrix} \frac{-1}{\beta\sigma} & \frac{1}{\beta\sigma} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\beta\sigma} & \frac{-1}{\beta\sigma} \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{q_y^2}{q_P} \begin{bmatrix} 0 & 0 & \frac{-1}{\beta\sigma} & \frac{1}{\beta\sigma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\text{and} \quad \mathbf{B} \equiv \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{\sigma} & \frac{-1}{\sigma} \\ 0 & 0 \end{bmatrix} + q_y \begin{bmatrix} \frac{-1}{\sigma} & \frac{1}{\sigma} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since this system has two predetermined variables ($\hat{m}_{t-1}$ and $\hat{y}_{t-1}$) and two non-predetermined variables ($\hat{m}_{t+1}$ and $\hat{y}_{t+1}$), the necessary and sufficient condition for local-equilibrium determinacy is that the matrix $\mathbf{A}$ have two eigenvalues inside the unit circle and two eigenvalues outside. We write the characteristic polynomial of $\mathbf{A}$ as

$$\det\left(\mathbf{A} - z\mathbf{I}_4\right) = z\mathcal{Q}\left(z\right),$$

where $\mathbf{I}_4$ denotes the 4×4 identity matrix and

$$\begin{aligned}\mathcal{Q}\left(z\right) \quad &\equiv \quad \mathcal{P}\left(z\right) - \left[\left(\frac{\delta_m \kappa}{\beta\sigma} + \frac{1}{\sigma\chi_i}\right) q_y - \frac{\delta_m \kappa}{\beta}\left(q_P - 1\right)\right] z^2 \\ &+ \left[\left(\frac{\delta_m \kappa}{\beta\sigma} + \frac{1+\beta}{\beta\sigma\chi_i}\right) q_y + \left(\frac{-\delta_m \kappa}{\beta} + \frac{\kappa}{\beta\sigma\chi_i} - \frac{\delta_m \chi_y \kappa}{\beta\sigma\chi_i}\right)\left(q_P - 1\right)\right] z - \frac{q_y}{\beta\sigma\chi_i},\end{aligned}$$

where in turn $\mathcal{P}\left(z\right)$ is defined in Appendix B.2. Thus, the eigenvalues of $\mathbf{A}$ are 0 and the roots of $\mathcal{Q}\left(z\right)$. We have

$$\mathcal{Q}\left(0\right) = \frac{-1}{\beta} - \frac{\chi_y}{\beta\sigma\chi_i} - \frac{q_y}{\beta\sigma\chi_i} < -1 \quad \text{and} \quad \mathcal{Q}\left(1\right) = \frac{\left(1 - \delta_m \chi_y\right)\kappa q_P}{\beta\sigma\chi_i} > 0,$$

where the second inequality follows from (B.11). Therefore, $\mathcal{Q}\left(z\right)$ has either one root or three roots in the real-number interval $(0,1)$. Now, the product of the three roots of $\mathcal{Q}\left(z\right)$ is equal to $-\mathcal{Q}\left(0\right) > 1$, so that $\mathcal{Q}\left(z\right)$ has at least one root outside the unit circle. As a consequence, $\mathcal{Q}\left(z\right)$ has exactly one root inside the real-number interval $(0,1)$.

The other roots of $\mathcal{Q}\left(z\right)$ are either two real numbers outside $[0,1]$, or two conjugate complex numbers. In the latter case, both are outside the unit circle, since $\mathcal{Q}\left(z\right)$ has at least one root outside it. Therefore, $\mathcal{Q}\left(z\right)$ has exactly two roots outside the unit circle if and only if it has no root inside the real-number interval $[-1,0)$. Since $\mathcal{Q}\left(0\right) < 0$, this condition is equivalent to $\mathcal{Q}\left(z\right) < 0$ for all $z \in [-1,0]$. Thus, the necessary and sufficient condition for determinacy is $\mathcal{Q}\left(z\right) < 0$ for all $z \in [-1,0]$.

A *sufficient* condition for determinacy is, therefore, that $\mathcal{Q}\left(z\right) < 0$ for all $z \in [-1,0]$ *and all* $\theta \in (0,1)$. To restate this sufficient condition, we rewrite $\mathcal{Q}(z)$ as

$$\mathcal{Q}\left(z\right) = \frac{\kappa}{\beta}\left[\mathcal{Q}_1\left(z\right) + \mathcal{Q}_2\left(z\right)\right],$$

where

$$\begin{aligned}\mathcal{Q}_1\left(z\right) \quad &\equiv \quad \frac{-1}{\kappa}\left(1 - z\right)\left(1 - \beta z\right)\left(1 + \frac{\chi_y + q_y}{\sigma\chi_i} - z\right), \\ \mathcal{Q}_2\left(z\right) \quad &\equiv \quad \left(\frac{1 + \delta_m q_y}{\sigma} - \delta_m q_P\right) z\left(1 - z\right) + \left(\frac{1 - \delta_m \chi_y}{\sigma\chi_i}\right) q_P z.\end{aligned}$$

The only reduced-form parameter that depends on the degree of price stickiness θ is the slope of the Phillips curve κ, which is decreasing in θ. Therefore, for any $z \in [-1,0]$, $\mathcal{Q}_1\left(z\right)$ is decreasing in θ, while $\mathcal{Q}_2\left(z\right)$ does not depend on θ. As a consequence, $\mathcal{Q}\left(z\right) < 0$ for all $z \in [-1,0]$ and all $\theta \in (0,1)$ if and only if $\mathcal{Q}\left(z\right) < 0$ for all $z \in [-1,0]$ as $\theta \to 0$. In turn, this condition is equivalent to $\mathcal{Q}_2(z) < 0$ for all $z \in [-1, 0)$, since $\lim_{\theta \to 0} \kappa = +\infty$ implies $\lim_{\theta \to 0} \mathcal{Q}_1(z) = 0$. Now, we can rewrite $\mathcal{Q}_2(z)$ as

$$\mathcal{Q}_2(z) = z\left\{Kz + \left[\left(\frac{1-\delta_m \chi_y}{\sigma \chi_i}\right) q_P - K\right]\right\},$$

where $K \equiv \delta_m q_P - (1 + \delta_m q_y)/\sigma$. Therefore, $\mathcal{Q}_2(z) < 0$ for all $z \in [-1, 0)$ if and only if

$$K < \left(\frac{1-\delta_m \chi_y}{2\sigma \chi_i}\right) q_P, \tag{C.4}$$

where the right-hand side is non-negative, as follows from the second inequality in (B.11).

Now consider the following condition:

$$q_P \leq \left\{\left(\frac{I^\ell - I}{I^\ell}\right)\left[\frac{-m\Gamma_{\ell m}(\ell, m)}{\Gamma_\ell(\ell, m)}\right]\right\}^{-1}. \tag{C.5}$$

This condition implies $\delta_m q_P \leq 1/\sigma$ because

$$\delta_m < \frac{\alpha_\ell \alpha_\phi}{\sigma}\left(\frac{-\Gamma_{\ell m} m}{\Gamma_\ell}\right) \leq \frac{\alpha_\ell}{\sigma}\left(\frac{-\Gamma_{\ell m} m}{\Gamma_\ell}\right) = \left(\frac{I^\ell - I}{I^\ell}\right)\left(\frac{-\Gamma_{\ell m} m}{\Gamma_\ell}\right)\frac{1}{\sigma}.$$

In turn, $\delta_m q_P \leq 1/\sigma$ implies $K < 0$, which in turn implies (C.4), which in turn implies that $\mathcal{Q}(z)$ has exactly two roots outside the unit circle, which finally implies determinacy. Therefore, Condition (C.5) is a sufficient condition for determinacy.

C.3 Assessment of the Sufficient Determinacy Condition

To assess whether Condition (C.5) is likely to be met or not, we proceed as follows. We consider, for simplicity, a Cobb-Douglas specification for the production function f^b:

$$f^b\left(h_t^b, m_t\right) \equiv A_b \left(h_t^b\right)^{1-\varsigma} (m_t)^\varsigma,$$

where $A_b > 0$ and $0 < \varsigma < 1$. This specification implies that the steady-state elasticity of marginal banking costs to reserves, which appears in Condition (C.5), can be rewritten as

$$\frac{-m\Gamma_{\ell m}(\ell, m)}{\Gamma_\ell(\ell, m)} = \left(\frac{\varsigma}{1-\varsigma}\right)\left[1 + \frac{\upsilon^{b\prime\prime}(h^b)\, h^b}{\upsilon^{b\prime}(h^b)}\right].$$

Our Cobb-Douglas specification for f^b also implies that households' first-order conditions (6) and (7), at the steady state, can be combined to get

$$\varsigma = \left(\frac{m}{\ell}\right)\left(\frac{I - I^m}{I^\ell - I}\right).$$

Therefore, Condition (C.5) can be rewritten as

$$q_P \leq \frac{1 - \left(\frac{m}{\ell}\right)\left(\frac{I - I^m}{I^\ell - I}\right)}{\left(\frac{m}{\ell}\right)\left(\frac{I - I^m}{I^\ell}\right)\left[1 + \frac{\upsilon^{b\prime\prime}(h^b) h^b}{\upsilon^{b\prime}(h^b)}\right]}. \tag{C.6}$$

We set the steady-state variables I^m, I^ℓ, and m/ℓ to match some features of the US economy during the 2008-2015 ZLB episode. This episode lasted from December 16, 2008, to December 16, 2015; because we use monthly data, however, we consider the period from January 2009 to November 2015. We set the net IOR rate $I^m - 1$ to 0.25% per annum (the constant value of the interest rate on excess reserves over the period); the net interest rate on bank loans $I^\ell - 1$ to 3.25% per annum (the average value of the bank prime loan rate over the period); and the ratio of bank reserves to loans m/ℓ to 0.18 (the average ratio of total reserves of depository institutions to bank credit of all commercial banks over the period).

We calibrate the parameter q_P to match the increase in the stock of nominal reserves over the period. More specifically, we rewrite (C.2) as $\hat{M}_t = (1 - q_P)\hat{P}_t - q_y \hat{y}_t$, and we assume conservatively that the Fed increased the stock of nominal reserves over the period only in response to low inflation, not in response to low output growth (i.e., $q_y = 0$). This assumption is conservative because it tends to overestimate q_P and, therefore, to make Condition (C.6) harder to satisfy. Under this assumption, we get

$$q_P = 1 - \frac{\hat{M}_t}{\hat{P}_t}. \tag{C.7}$$

Our model has constant reserves and prices at the steady state. In reality, however, reserves followed a positive trend before the crisis, and the Fed has a positive inflation target. The Fed increased the stock of nominal reserves *beyond its trend*, in response to inflation *below target*. A natural empirical counterpart of (C.7), over the January 2009-November 2015 period, is therefore

$$q_P = 1 - \frac{(\ln M_{2015:11} - \ln M_{2009:01}) - \Delta_M}{(\ln P_{2015:11} - \ln P_{2009:01}) - \Delta_P},$$

where M_t and P_t are measured by the total reserves of depository institutions and the consumer price index respectively, while Δ_M and Δ_P denote respectively the "neutral" trend growth rate in reserves and the targeted growth rate in prices over the January 2009-November 2015 period. We conservatively set Δ_M to zero, thus attributing all of the observed growth in reserves to the Fed's response to inflation below target. Like the assumption $q_y = 0$, the assumption $\Delta_M = 0$ is conservative because it tends to overestimate q_P and, therefore, to make Condition (C.6) harder to satisfy. And we set Δ_P to 14%, which corresponds to the Fed's 2% annual-inflation target over (almost) seven years. We then get $q_P = 50.0$.

Finally, we make conservative assumptions about the values of the steady-state variables $h^b v^{b\prime\prime}(h^b)/v^{b\prime}(h^b)$ and I. More specifically, we set $h^b v^{b\prime\prime}(h^b)/v^{b\prime}(h^b)$, the inverse of the steady-state Frisch elasticity of bankers' labor supply, to 5. The value 5 for the inverse of a Frisch elasticity of labor supply lies at the upper end of the range of microeconomic estimates, and is much higher than values commonly considered in macroeconomics. And we set the net interest rate $I - 1$ to 0.75% per annum, i.e. 50 basis points per annum above the net IOR rate $I^m - 1$. This value is much higher than the average value, over the January 2009-November 2015 period, of standard proxies for $I - 1$, like the 3-month T-bill rate or the 3-month AA (financial or non-financial) commercial paper rate. Our assumptions about $h^b v^{b\prime\prime}(h^b)/v^{b\prime}(h^b)$ and I are conservative because they tend to overestimate these two steady-state variables and, therefore, to make Condition (C.6) harder to satisfy.

The right-hand side of Condition (C.6) depends on the period length, through the ratio $(I - I^m)/I^\ell$. Since one-period bank loans in our model are working-capital loans, which are short-term loans in reality, we set the period length to one quarter. Thus, we express all the interest rates in the ratio $(I - I^m)/I^\ell$ as quarterly rates.

We then get the value 705.7 for the right-hand side of Condition (C.6). This value is one order of magnitude larger than the value 50.0 obtained for the left-hand side of Condition (C.6). We thus find that Condition (C.6) is met by a large margin even under our conservative assumptions. We conclude that setting exogenously the IOR rate and following the reserves-supply rule still delivers local-equilibrium determinacy, except for implausible calibrations.

Appendix D: Extended Model With Household Cash

In the main text (Section 3), we show that our benchmark model delivers local-equilibrium determinacy under exogenous monetary-policy instruments, and we use this determinacy result to explain the low volatility of inflation and the absence of significant deflation at the ZLB. Our benchmark model, however, is specific in that households hold money only in the form of reserves, in their capacity as bankers. This makes our point stark because banks cannot collectively change the aggregate nominal quantity of reserves outstanding. In reality, bank reserves can fall if households demand more cash. In the present appendix, we show that our results do not unravel when we allow for such leakages out of reserve balances. More specifically, we introduce household cash into our benchmark model through a cash-in-advance constraint; we derive a sufficient condition for local-equilibrium determinacy in the resulting model, under an exogenous IOR rate and an exogenous monetary base (made of bank reserves and household cash); and we argue that this sufficient determinacy condition is likely to be met.

D.1 Introducing Household Cash into the Benchmark Model

We assume that each period is made of a financial exchange followed by a goods exchange. Households acquire cash in the financial exchange and use it to buy goods in the goods exchange; firms receive this cash in the goods exchange and have to wait until the next period's financial exchange to spend it (repaying loans). Thus, households choose bonds b_t, consumption c_t, work hours h_t, loans ℓ_t, reserves m_t, and (now) cash m_t^c to maximize the same reduced-form utility function (1) as previously, subject to the budget constraint

$$m_t^c + b_t + \ell_t + m_t \leq \frac{m_{t-1}^c - c_{t-1}}{\Pi_t} + \frac{I_{t-1}}{\Pi_t} b_{t-1} + \frac{I_{t-1}^\ell}{\Pi_t} \ell_{t-1} + \frac{I_{t-1}^m}{\Pi_t} m_{t-1} + w_t h_t + \omega_t$$

and the cash-in-advance constraint

$$m_t^c \geq c_t, \tag{D.1}$$

taking all prices (I_t, I_t^ℓ, I_t^m, P_t, and w_t) as given. Letting λ_t and λ_t^c denote the Lagrange multipliers on these two constraints respectively, the first-order conditions of households' optimization problem are again (3), (4), (5), (6), (7), and now

$$\lambda_t^c + \frac{\beta\lambda_{t+1}}{\Pi_{t+1}} - \lambda_t = 0.$$

The objective of firm i is now to maximize

$$\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \frac{\beta\lambda_{t+k+1}}{\lambda_t \Pi_{t,t+k+1}} \left[P_t^* (i) \, y_{t+k} (i) - I_{t+k}^\ell L_{t+k}(i) - [W_{t+k} h_{t+k} (i) - L_{t+k}(i)] \right] \right\},$$

since the firm has to wait until the next period to exchange its cash. The first-order condition for the firm's optimization problem is thus

$$\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \frac{\beta\lambda_{t+k+1}}{\lambda_t \Pi_{t,t+k+1}} \left[P_t^* (i) - \left(\frac{\varepsilon}{\varepsilon - 1} \right) \left(\phi I_{t+k}^\ell + (1 - \phi) \right) \frac{W_{t+k}}{f' [h_{t+k} (i)]} \right] y_{t+k} (i) \right\} = 0, \tag{D.2}$$

instead of (12). In the particular case of flexible prices (and in a symmetric equilibrium), this first-order condition becomes

$$P_t = \frac{\varepsilon}{\varepsilon - 1} \left[\phi I_t^\ell + (1 - \phi) \right] \frac{W_t}{f' (h_t)}, \tag{D.3}$$

which replaces (13). None of the other equilibrium conditions stated in Section 2 is changed, except the reserve-market-clearing condition (16), which is replaced by the money-market-clearing condition

$$m_t + m_t^c = \frac{M_t}{P_t}, \tag{D.4}$$

since the monetary base M_t controlled by the central bank is now made not only of bank reserves, but also of household cash. As previously, the equilibrium conditions (3), (5), (9), (10) holding with equality, and (17) imply the relationship (18) between loans and employment.

D.2 Steady State and Log-Linearization

To derive the necessary and sufficient condition for steady-state existence and uniqueness under a constant IOR rate ($I_t^m = I^m \geq 1$), a constant monetary base ($M_t = M > 0$), and no discount-factor shocks ($\zeta_t = 1$), we first note that the steady-state inflation rate Π is equal to 1 under a constant monetary base. In turn, $\Pi = 1$ and (4) together imply that the steady-state interest rate on bonds I is equal to $1/\beta$, as previously. Using (3), (5), (9), (17), (18), (D.3), and $I = 1/\beta$, we can rewrite households' first-order condition for loans (6) at the steady state as a relationship between real reserves m and employment h:

$$\Gamma_\ell [\mathcal{L} (h), m] = \tilde{\mathcal{A}} (h) \equiv u' [f (h)] \left\{ \frac{\beta}{\phi} \left[\left(\frac{\varepsilon - 1}{\varepsilon} \right) \frac{u' [f (h)] f' (h)}{v' (h)} - (1 - \phi) \right] - 1 \right\}. \tag{D.5}$$

Because the left-hand side of (D.5) is positive, we restrict the domain of the function $\tilde{\mathcal{A}}$ to $(0, \tilde{h})$, where $\tilde{h} > 0$ is implicitly and uniquely defined by $u'[f(\tilde{h})]f'(\tilde{h})/v'(\tilde{h}) = [\phi/\beta + (1-\phi)]\varepsilon/(\varepsilon - 1)$. The function $\tilde{\mathcal{A}}$ is strictly decreasing ($\tilde{\mathcal{A}}' < 0$), with $\lim_{h\to 0} \tilde{\mathcal{A}}(h) = +\infty$ and $\lim_{h\to\tilde{h}} \tilde{\mathcal{A}}(h) = 0$.

Since $\Gamma_{\ell\ell} > 0$, $\mathcal{L}' > 0$, $\Gamma_{\ell m} < 0$, and $\tilde{\mathcal{A}}' < 0$, Equation (D.5) implicitly and uniquely defines a function $\tilde{\mathcal{M}}$ which is strictly increasing ($\tilde{\mathcal{M}}' > 0$) and such that

$$m = \tilde{\mathcal{M}}(h). \tag{D.6}$$

Using (3), (9), (17), (18), (D.6), and $I = 1/\beta$, we can then rewrite households' first-order condition for reserves (7) at the steady state as

$$\tilde{\mathcal{F}}(h) \equiv \frac{\Gamma_m \left[\mathcal{L}(h), \tilde{\mathcal{M}}(h)\right]}{u'[f(h)]} = -(1 - \beta I^m).$$

The same reasoning as in Appendix A.5, this time applied to $\tilde{\mathcal{F}}$ instead of $\mathcal{F}$, shows that the function $\tilde{\mathcal{F}}$ is strictly increasing from $-\infty$ to 0. Therefore, the necessary and sufficient condition for existence and uniqueness of a steady state is again $I^m < 1/\beta$.

Log-linearizing the model around its unique steady state, we get the same IS equation (21) and reserves-demand equation (23) as previously. Using the goods-market-clearing condition (17) and the binding cash-in-advance constraint (D.1) to rewrite the money-market-clearing condition (D.4), and then log-linearizing the resulting equation, leads to

$$\hat{M}_t - \hat{P}_t = (1 - \alpha_c)\hat{m}_t + \alpha_c \hat{y}_t, \tag{D.7}$$

where $\alpha_c \equiv f(h)/[f(h) + \mathcal{M}(h)] \in (0,1)$ denotes the steady-state share of household cash in the monetary base. Finally, we derive the Phillips curve by following the same steps as in Appendix B.1. More specifically, the log-linearized first-order condition is now

$$\hat{P}_t^* = (1 - \beta\theta)\,\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \left(\alpha_\phi i_{t+k}^{\ell} + \hat{w}_{t+k} + \hat{P}_{t+k} - \widehat{mp}_{t+k|t} \right) \right\},$$

which corresponds to (B.1) without the i_{t+k} term; and this log-linearized first-order condition leads to the Phillips curve

$$\pi_t = \beta \mathbb{E}_t \{\pi_{t+1}\} + \kappa\left(\hat{y}_t - \delta_m \hat{m}_t + \delta_i i_t\right) \tag{D.8}$$

with $\delta_i \equiv \alpha_\phi \Psi/\kappa > 0$, where the term $\delta_i i_t$ captures the opportunity cost for firms of holding their cash from one period to the next.

D.3 Derivation of a Sufficient Determinacy Condition

Using the IS equation (21), the reserves-demand equation (23), the money-market-clearing condition (D.7), the Phillips curve (D.8), and the identity $\pi_t = \hat{P}_t - \hat{P}_{t-1}$, we get the following dynamic system in $\hat{m}_t$ and $\hat{y}_t$:

$$\mathbb{E}_t \left\{ \begin{bmatrix} \hat{m}_{t+1} \\ \hat{m}_t \\ \hat{y}_{t+1} \\ \hat{y}_t \end{bmatrix} \right\} = \tilde{\mathbf{A}} \begin{bmatrix} \hat{m}_t \\ \hat{m}_{t-1} \\ \hat{y}_t \\ \hat{y}_{t-1} \end{bmatrix} + \mathbb{E}_t \left\{ \tilde{\mathbf{B}} \begin{bmatrix} i_t^m \\ \hat{\mu}_{t+1} \\ \hat{\mu}_t \\ r_t \end{bmatrix} \right\}, \tag{D.9}$$

where $\hat{\mu}_t = \hat{M}_t - \hat{M}_{t-1}$ denotes the log-deviation of the gross growth rate of nominal reserves $\mu_t \equiv M_t/M_{t-1}$ from its steady-state value 1,

$$\tilde{\mathbf{A}} \equiv \begin{bmatrix} \frac{1+\beta-\delta_m\kappa}{\beta} - \frac{\delta_i\kappa}{\beta\chi_i} & \frac{-1}{\beta} & \frac{\kappa}{\beta} + \frac{\delta_i\chi_y\kappa}{\beta\chi_i} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1-\delta_m\kappa}{\beta\sigma} - \frac{1}{\sigma\chi_i} - \frac{\delta_i\kappa}{\beta\chi_i\sigma} & \frac{-1}{\beta\sigma} & 1 + \frac{\chi_y}{\sigma\chi_i} + \frac{\kappa}{\beta\sigma} + \frac{\delta_i\chi_y\kappa}{\beta\chi_i\sigma} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \alpha_c \begin{bmatrix} \frac{-1}{\beta\sigma} & \frac{1}{\beta\sigma} & \frac{1}{\beta\sigma} & \frac{-1}{\beta\sigma} \\ 0 & 0 & 0 & 0 \\ \frac{-1}{\beta\sigma} & \frac{1}{\beta\sigma} & \frac{1}{\beta\sigma} & \frac{-1}{\beta\sigma} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+\frac{\alpha_c}{1-\alpha_c} \begin{bmatrix} \frac{1}{\sigma\chi_i} + \frac{(1-\sigma)\delta_m\kappa}{\beta\sigma} + \frac{(1-\sigma)\delta_i\kappa}{\beta\chi_i\sigma} & 0 & \frac{-(1-\sigma)}{\beta\sigma} - \frac{\chi_y}{\sigma\chi_i} - \frac{(1-\sigma)\kappa}{\beta\sigma} - \frac{(1-\sigma)\delta_i\chi_y\kappa}{\beta\chi_i\sigma} & \frac{1-\sigma}{\beta\sigma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{and} \quad \tilde{\mathbf{B}} \equiv \begin{bmatrix} \frac{\delta_i\kappa}{\beta} & 1 & \frac{-1}{\beta} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\sigma} + \frac{\delta_i\kappa}{\beta\sigma} & 0 & \frac{-1}{\beta\sigma} & \frac{-1}{\sigma} \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\alpha_c}{1-\alpha_c} \begin{bmatrix} \frac{-1}{\sigma} + \frac{(1-\sigma)\delta_i\kappa}{\beta\sigma} & 1 & \frac{1-\sigma}{\beta\sigma} & \frac{1}{\sigma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since this system has two predetermined variables ($\hat{m}_{t-1}$ and $\hat{y}_{t-1}$) and two non-predetermined variables ($\hat{m}_{t+1}$ and $\hat{y}_{t+1}$), the necessary and sufficient condition for local-equilibrium determinacy is that the matrix $\tilde{\mathbf{A}}$ have two eigenvalues inside the unit circle and two eigenvalues outside. We note that $\tilde{\mathbf{A}}$ can be obtained from $\mathbf{A}$ by replacing q_P, q_y, κ, and δ_m by respectively $1/(1-\alpha_c)$, $\alpha_c/(1-\alpha_c)$, $\tilde{\kappa} \equiv (1+\delta_i\chi_y/\chi_i)\kappa$, and $\tilde{\delta}_m \equiv (\delta_m\chi_i+\delta_i)/(\chi_i+\delta_i\chi_y)$ in $\mathbf{A}$. Therefore, we deduce from Appendix C.2 that $\tilde{\mathbf{A}}$ has two eigenvalues inside the unit circle and two eigenvalues outside for any $\theta \in (0,1)$ if and only if

$$\tilde{K} \equiv \frac{-1}{\sigma} + \left[1 + \left(\frac{\alpha_c}{1-\alpha_c}\right)\left(\frac{\sigma-1}{\sigma}\right)\right]\tilde{\delta}_m < \frac{1-\tilde{\delta}_m\chi_y}{2\sigma\chi_i\,(1-\alpha_c)}, \tag{D.10}$$

where the right-hand side is positive since $1-\tilde{\delta}_m\chi_y = (1-\delta_m\chi_y)\chi_i/(\chi_i+\delta_i\chi_y) > 0$ (as follows from the second inequality in (B.11)). We have

$$\begin{aligned} \tilde{K} &< \frac{-1}{\sigma} + \frac{\tilde{\delta}_m}{1-\alpha_c} \\ &< \frac{-1}{\sigma} + \frac{1}{1-\alpha_c}\left(\delta_m + \frac{\delta_i}{\chi_i}\right) \\ &< \frac{1}{\sigma}\left[-1 + \frac{1}{1-\alpha_c}\left(-\alpha_\ell\frac{\Gamma_{\ell m}m}{\Gamma_\ell} - \alpha_m\frac{\Gamma_{mm}m}{\Gamma_m}\right)\right] \\ &\leq \bar{K} \equiv \frac{1}{\sigma}\left[-1 + \frac{1}{1-\alpha_c}\left(-\alpha_\ell\frac{\Gamma_{\ell m}m}{\Gamma_\ell} + \alpha_m\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\right)\right], \end{aligned}$$

where the last inequality follows from (B.14). In turn, using first (6) and (7), and then (10) with equality and (D.3), we get sequentially

$$\begin{aligned}
\bar{K} &= \frac{1}{\sigma}\left[-1+\left(\frac{1-\beta I^m}{1-\alpha_c}\right)\left(\frac{I}{I^\ell}\frac{\Gamma_{\ell m}m}{\Gamma_m}+\frac{1}{\beta I^m}\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\right)\right] \\
&= \frac{1}{\sigma}\left\{-1+\left(\frac{1-\beta I^m}{1-\alpha_c}\right)\left[\left[\left(\frac{\varepsilon-1}{\varepsilon}\right)\left(\frac{f'h}{f}\right)-\left(\frac{1-\phi}{\phi}\right)\left(\frac{\ell}{y}\right)\right]^{-1}\left(\frac{m}{\beta y}\right)+\frac{1}{\beta I^m}\right]\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\right\} \\
&\leq \frac{1}{\sigma}\left\{-1+\left(\frac{1-\beta I^m}{1-\alpha_c}\right)\left[\frac{1}{\beta}\left(\frac{\varepsilon}{\varepsilon-1}\right)\left(\frac{f}{f'h}\right)\left(\frac{m}{y}\right)+\frac{1}{\beta I^m}\right]\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\right\}.
\end{aligned}$$

Now consider the following condition, which states that the last expression is negative:

$$\left(\frac{1-\beta I^m}{1-\alpha_c}\right)\left\{\frac{1}{\beta}\left(\frac{\varepsilon}{\varepsilon-1}\right)\left[\frac{f(h)}{hf'(h)}\right]\left(\frac{m}{y}\right)+\frac{1}{\beta I^m}\right\}\left[\frac{\ell\Gamma_{\ell m}(\ell,m)}{\Gamma_m(\ell,m)}\right]<1. \tag{D.11}$$

This condition implies $\bar{K}<0$, which in turn implies $\tilde{K}<0$, which in turn implies (D.10), which in turn implies that $\tilde{\mathbf{A}}$ has two eigenvalues inside the unit circle and two eigenvalues outside, which finally implies determinacy.

D.4 Assessment of the Sufficient Determinacy Condition

To assess whether Condition (D.11) is likely to be met or not, we proceed broadly along the same lines as in Appendix C.3. We consider, for simplicity, a Cobb-Douglas specification for the production function f^b:

$$f^b\left(h_t^b,m_t\right)\equiv A_b\left(h_t^b\right)^{1-\varsigma}(m_t)^{\varsigma},$$

where $A_b>0$ and $0<\varsigma<1$. This specification implies that the steady-state elasticity of Γ_m that appears in Condition (D.11) can be rewritten as

$$\frac{\ell\Gamma_{\ell m}(\ell,m)}{\Gamma_m(\ell,m)}=\left(\frac{1}{1-\varsigma}\right)\left[1+\frac{v^{b\prime\prime}\left(h^b\right)h^b}{v^{b\prime}\left(h^b\right)}\right].$$

Our Cobb-Douglas specification for f^b also implies that households' first-order conditions (6) and (7), at the steady state, can be combined to get

$$\varsigma=\left(\frac{m}{\ell}\right)\left(\frac{I-I^m}{I^\ell-I}\right).$$

Therefore, Condition (D.11) can be rewritten as

$$\left(\frac{I-I^m}{1-\alpha_c}\right)\left\{\left(\frac{\varepsilon}{\varepsilon-1}\right)\left[\frac{f(h)}{hf'(h)}\right]\left(\frac{m}{y}\right)+\frac{1}{I^m}\right\}\left[\frac{1+\frac{v^{b\prime\prime}(h^b)h^b}{v^{b\prime}(h^b)}}{1-\left(\frac{m}{\ell}\right)\left(\frac{I-I^m}{I^\ell-I}\right)}\right]<1. \tag{D.12}$$

We set the steady-state variables I^m, I^ℓ, m/ℓ, α_c, and m/y to match some features of the US economy during the 2008-2015 ZLB episode. More specifically, as in Appendix C.3, we set the net interest rates I^m-1 and $I^\ell-1$ to 0.25% and 3.25% per annum respectively, and the ratio

m/ℓ to 0.18. We set the steady-state share of household cash in the monetary base, α_c, to 0.39, which is the average value of the ratio between the currency component of M1 and the monetary base from January 2009 to November 2015. And we set the ratio m/y to 0.40, which is the average value of the ratio between total reserves of depository institutions and quarterly GDP from 2009Q1 to 2015Q4.

We make standard assumptions about the steady-state elasticity of output to labor $hf'(h)/f(h)$ and the elasticity of substitution between goods ε. More specifically, we set the former to 0.66, and the latter to 6 (implying a 20% markup).

Finally, we make the same conservative assumptions as in Appendix C.3 about the values of the steady-state variables $h^b v^{b\prime\prime}(h^b)/v^{b\prime}(h^b)$ and I. More specifically, we set the elasticity $h^b v^{b\prime\prime}(h^b)/v^{b\prime}(h^b)$ to 5, and the net interest rate $I-1$ to 0.75% per annum. These assumptions are conservative because they tend to overestimate these two steady-state variables and, therefore, to make Condition (D.12) harder to satisfy.

The left-hand side of Condition (D.12) depends on the period length through the ratio $(I - I^m)/I^m$ (and only through this ratio, since the ratio $(I - I^m)/y$ does not depend on the period length). We set the period length to one quarter, as in Appendix C.3. Thus, we express all the interest rates in the ratio $(I - I^m)/I^m$ as quarterly rates.

We then get the value 0.02 for the left-hand side of Condition (D.12). This value is one to two orders of magnitude smaller than 1. We thus find that Condition (D.12) is met by a large margin even under our conservative assumptions. We conclude that the introduction of household cash into the monetary base does not affect the ability of our model to deliver local-equilibrium determinacy under an exogenous IOR rate and an exogenous monetary base, except for implausible calibrations.

Appendix E: Extended Model With Liquid Government Bonds

In this appendix, we motivate, present and analyze our extended model with liquid government bonds. We show that this model can account not only for the three key features of inflation during ZLB episodes (like our benchmark model), but also for the negative spread between T-bill and IOR rates observed during these episodes (unlike our benchmark model). More specifically, we show that this model has an equilibrium in which the return on government bonds is below the IOR rate, while demand for bank reserves is not satiated. This equilibrium coincides with the equilibrium of our benchmark model (without liquid bonds), in the sense that all the endogenous variables that are common to both models, except the lump-sum transfer T_t, take the same equilibrium values. We show the existence of this equilibrium under two simple parameter restrictions, and we show how one of these parameter restrictions can be relaxed without affecting our results.

E.1 Satiation vs. Non-Satiation of Demand for Reserves

In Sections 3 and 4, we have shown that our benchmark model can broadly account for three key observations about US inflation during ZLB episodes (no significant deflation, little inflation volatility, and no significant inflation following QE policies). These results rest on the assumption that demand for bank reserves got close to satiation, but did not reach full satiation, i.e. that bank reserves carried a small but positive convenience yield ($I_t^m < I_t$ and $\Gamma_m > 0$). If we allowed for a finite satiation point in the demand for reserves and if demand for reserves were fully satiated ($I_t^m = I_t$ and $\Gamma_m = 0$), then our results would fall apart as follows: (i) in the local analysis of Section 3, our model would be isomorphic to the basic NK model ($\delta_m = 0$ and $i_t = i_t^m$), and would generate indeterminacy under a permanent interest-rate peg; and (ii) without price-level determinacy, the numerical simulation of QE2 in Section 4 would not be possible.

As we noted in the Introduction, our non-satiation assumption stands in contrast to views often expressed about the US economy in recent years. One argument making a case for satiation of demand for reserves is the fact that the second and third rounds of quantitative easing (QE2 and QE3) had no apparent inflationary consequences, as Reis (2016) and Cochrane (2018) point out. On this front, our counter-argument is simply that this fact may also be consistent with demand for reserves being close to satiation, rather than fully satiated, as our numerical simulation of QE2 in Section 4 suggests.

In the present appendix, we address a second argument that goes against our non-satiation view. This argument is the fact that T-bill returns were often below the IOR rate during ZLB episodes. We do not think this fact contradicts our claim that reserves still had a positive marginal convenience yield during this period. The lower T-bill returns, we argue, could reflect strong demand by non-bank entities – using T-bills as collateral or international reserve asset, for instance. We formalize our counter-argument by introducing government bonds providing liquidity services into our benchmark model. We show that our model with liquid bonds has an equilibrium in which the return on government bonds is below the IOR rate. Moreover, this equilibrium of our model with liquid bonds coincides with the equilibrium of our benchmark model (without liquid bonds), in the sense that all the endogenous variables that are common to both models, except the lump-sum transfer T_t, take the same equilibrium values. So, all the results that we have obtained in our benchmark model in Sections 3 and 4 still hold in our model with liquid bonds.

E.2 Liquidity of Government Bonds

Our benchmark model abstracts from government bonds and any role they may play in facilitating transactions. In reality, banks may hold government bonds (or other liquid assets), in addition to reserves, for liquidity management. Some regulatory constraints that give rise to a convenience yield for reserves – like the constraint on "high-quality liquid assets" imposed on US banks – can also be satisfied by holding government bonds. From this (regulatory) vantage point, bonds and reserves are perfect substitutes in satisfying liquidity needs. But government bonds are not as useful as reserves in satisfying the intra-day liquidity needs that arise from banking transactions, because bonds can either be sold for next-day settlement or used in repo transactions arranged to obtain liquidity, while reserves are readily available for any transaction – as Bush et al. (2019) elaborate.

Government bonds also provide a convenience yield to many non-bank entities (e.g. by serving as collateral or international reserve asset) and benefit from regulations (like restrictions on the asset portfolios of US money-market mutual funds). So, the observed returns on government bonds may reflect their convenience yield. If the returns are sufficiently attractive compared to the IOR rate, banks may hold government bonds to satisfy liquidity needs and regulatory constraints. If not, banks may hold mostly reserves for liquidity management.

Our model abstracts from non-bank financial institutions and foreign entities that may hold bonds. To formalize our main point, we will assume that workers get utility from government bonds (instead of modeling, say, a pension fund that holds bonds on workers' behalf). We will show that bankers may use government bonds for liquidity management if the IOR rate is sufficiently low compared to the equilibrium return from holding liquid bonds; but bankers will only use reserves for liquidity management when the IOR rate is sufficiently high. Although we don't explicitly model inside assets like federal-funds loans, we have in mind that our equilibrium with a relatively high IOR rate can also represent observed episodes in which banks don't lend federal funds, and the federal-funds rate is below the IOR rate. Our main point is that financial institutions that don't have direct access to the IOR rate may hold these assets in equilibrium, while banks hold reserves with a positive marginal convenience yield.[3]

E.3 Equilibrium Conditions Related to Households

As in our benchmark model (presented in Section 2), the representative household consists of workers and bankers, and gets utility from consumption (c_t) and disutility from labor (h_t for workers, h_t^b for bankers). We now assume that workers also get utility from holding government bonds (b_t^w), and bankers may use government bonds (b_t^b) as well as reserves (m_t) to produce loans (ℓ_t). As before, we make a substitution for h_t^b in households' primitive utility function and get the following reduced-form utility function:

$$U_t = \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} \beta^k \zeta_{t+k} \left[u\left(c_{t+k}\right) - v\left(h_{t+k}\right) - \Gamma\left(\ell_{t+k}, m_{t+k} + \eta b_{t+k}^b\right) + z\left(b_{t+k}^w\right) \right] \right\},$$

[3]Bech and Klee (2012) present a model of segmentation in the federal-funds market in which limited access to the IOR rate (and other restrictions on trading) push the federal-funds rate below the IOR rate.

where $\eta \in (0,1]$. The function z, defined over $\mathbb{R}_{>0}$, is twice differentiable, strictly increasing ($z' > 0$), and strictly concave ($z'' < 0$); it also satisfies the usual Inada conditions. Values of η below unity may capture the fact that in reality reserves are more useful than government bonds for liquidity management because they provide immediate intra-day liquidity to banks (as discussed above). We allow for $\eta < 1$ to show that T-bill returns can be below the IOR rate even when T-bills provide smaller liquidity services than reserves to banks.

In the interest of realism (to make sure some reserves are always held in equilibrium), we also assume that the central bank imposes reserve requirements on banks. Since our model consolidates bankers and workers into households (thus, abstracting from deposits), we specify the reserve requirement as

$$m_t \geq \psi \ell_t, \tag{E.1}$$

where $\psi > 0$. The household budget constraint, expressed in real terms, is

$$c_t + b_t + b_t^b + b_t^w + \ell_t + m_t \leq \frac{I_{t-1}}{\Pi_t} b_{t-1} + \frac{I_{t-1}^b}{\Pi_t}\left(b_{t-1}^b + b_{t-1}^w\right) + \frac{I_{t-1}^\ell}{\Pi_t}\ell_{t-1} + \frac{I_{t-1}^m}{\Pi_t} m_{t-1} + w_t h_t + \tau_t,$$

where I_t^b denotes the gross nominal interest rate on government bonds (and b_t represents a private bond in zero net supply, as we indicated in Section 2). We let λ_t and λ_t^r denote the Lagrange multipliers on the period-t budget constraint and reserve requirement (respectively). The optimality conditions are

$$\lambda_t = \zeta_t u'(c_t),$$

$$\lambda_t w_t = \zeta_t v'(h_t),$$

$$\lambda_t = \beta I_t \mathbb{E}_t \left\{\frac{\lambda_{t+1}}{\Pi_{t+1}}\right\}, \tag{E.2}$$

$$\lambda_t = \zeta_t z'(b_t^w) + \beta I_t^b \mathbb{E}_t \left\{\frac{\lambda_{t+1}}{\Pi_{t+1}}\right\}, \tag{E.3}$$

$$\zeta_t \Gamma_\ell\left(\ell_t, m_t + \eta b_t^b\right) + \lambda_t + \psi \lambda_t^r = \beta I_t^\ell \mathbb{E}_t \left\{\frac{\lambda_{t+1}}{\Pi_{t+1}}\right\}, \tag{E.4}$$

$$\zeta_t \Gamma_m\left(\ell_t, m_t + \eta b_t^b\right) + \lambda_t = \lambda_t^r + \beta I_t^m \mathbb{E}_t \left\{\frac{\lambda_{t+1}}{\Pi_{t+1}}\right\}, \tag{E.5}$$

and

$$(m_t - \psi \ell_t)\lambda_t^r = 0.$$

We must also have

$$\eta \zeta_t \Gamma_m\left(\ell_t, m_t + \eta b_t^b\right) + \lambda_t \geq \beta I_t^b \mathbb{E}_t \left\{\frac{\lambda_{t+1}}{\Pi_{t+1}}\right\} \tag{E.6}$$

and $b_t^b \geq 0$, with complementary slackness.

E.4 Other Equilibrium Conditions

The remaining equilibrium conditions involve minor adjustments to our presentation in Subsections 2.2-2.4 (for firms, the government, and market clearing), as we describe below. The equilibrium conditions associated with firms don't change. For the government, we replace the consolidated budget constraint (14) by

$$M_t + B_t = I^m_{t-1} M_{t-1} + I^b_{t-1} B_{t-1} + T_t, \tag{E.7}$$

where B_t denotes the nominal stock of one-period public debt (held outside the central bank). A Ricardian fiscal policy adjusts the lump-sum transfer T_t to stabilize the real public debt around a steady-state target $b^* > 0$. The market-clearing conditions of Subsection 2.4 still apply − in particular the condition (15), given that private bonds are in zero net supply. In addition, we now have the market-clearing condition for government bonds:

$$b^w_t + b^b_t = \frac{B_t}{P_t}. \tag{E.8}$$

E.5 Characterization of our Equilibrium of Interest

The household optimality conditions above admit a solution with $1 = I^m_t < I^b_t$ that may represent the period before interest payment on reserves in the US (i.e. before 2008). If η is large enough, such a solution may have binding reserve requirements and $b^b_t > 0$. In this case, banks use government bonds − and if we extended our model, they could use inside assets like commercial paper − to manage the liquidity of their portfolios.

We are, however, interested in a candidate equilibrium in which banks do not use government bonds ($b^b_t = 0$), the reserve requirement is not binding ($\lambda^r_t = 0$), and the IOR rate is above the government-bond yield ($I^m_t > I^b_t$). This candidate equilibrium may capture − admittedly, in a stark way − some features of US bank portfolios and T-bill returns in the aftermath of the financial crisis.

In Appendix E.6, we prove the existence of such an equilibrium under two parameter restrictions. The first restriction is that the minimal reserves-to-loans ratio imposed by the central bank, ψ, should be lower than the steady-state value taken by the reserves-to-loans ratio in our benchmark model. This restriction is necessary for the reserve requirement (E.1) to be slack ($\lambda^r_t = 0$). The second restriction is that the steady-state marginal liquidity service of government bonds, $z'(b^*)$, should be large enough for the interest rate on government bonds to be lower than the IOR rate ($I^b_t < I^m_t$).

We now show that this equilibrium of interest (with $b^b_t = \lambda^r_t = 0$ and $I^b_t < I^m_t$) coincides with the equilibrium of our benchmark model, in the sense that all the endogenous variables that are common to both models, except the lump-sum transfer T_t, take the same equilibrium values.

Intuitively, in this equilibrium, banks hold only reserves ($b_t^b = 0$) because they pay more interest than government bonds ($I_t^b < I_t^m$) and are at least as liquid as government bonds ($\eta \leq 1$); and since banks do not hold government bonds ($b_t^b = 0$) and face a non-binding reserve requirement ($\lambda_t^r = 0$), they behave in exactly the same way as in our benchmark model.

To show that our equilibrium of interest (in the model with liquid bonds) coincides with the equilibrium of our benchmark model (without liquid bonds), we first use (E.2) to rewrite households' optimality conditions (E.3), (E.4), (E.5), and (E.6) in the following simpler forms:

$$\frac{I_t^b}{I_t} = 1 - \frac{\zeta_t z'\left(b_t^w\right)}{\lambda_t}, \tag{E.9}$$

$$\frac{I_t^\ell}{I_t} = 1 + \frac{\zeta_t \Gamma_\ell\left(\ell_t, m_t + \eta b_t^b\right)}{\lambda_t} + \psi \frac{\lambda_t^r}{\lambda_t}, \tag{E.10}$$

$$\frac{I_t^m}{I_t} = 1 + \frac{\zeta_t \Gamma_m\left(\ell_t, m_t + \eta b_t^b\right)}{\lambda_t} - \frac{\lambda_t^r}{\lambda_t}, \tag{E.11}$$

and

$$\frac{I_t^b}{I_t} \leq (1 - \eta) + \eta \frac{I_t^m}{I_t} + \eta \frac{\lambda_t^r}{\lambda_t}. \tag{E.12}$$

In our equilibrium of interest, because $b_t^b = \lambda_t^r = 0$, the equilibrium conditions (E.10) and (E.11) collapse to

$$\frac{I_t^\ell}{I_t} = 1 + \frac{\zeta_t \Gamma_\ell\left(\ell_t, m_t\right)}{\lambda_t} \quad \text{and} \quad \frac{I_t^m}{I_t} = 1 + \frac{\zeta_t \Gamma_m\left(\ell_t, m_t\right)}{\lambda_t}.$$

These conditions are identical to the equilibrium conditions (6) and (7) of our benchmark model. Therefore, our equilibrium of interest satisfies all the equilibrium conditions of our benchmark model (listed in Section 2), except the government budget constraint (14), which is replaced by (E.7). As a consequence, all the endogenous variables that are present in both models take the same values in our equilibrium of interest as in the equilibrium of our benchmark model, except the lump-sum transfer T_t appearing in the government budget constraint. So, we conclude that the introduction of liquid government bonds into our benchmark model enables us to account for the negative spread between Treasury-bill and IOR rates observed during ZLB episodes, without affecting in any way the ability of the model to account for the three key features of inflation during ZLB episodes (i.e., without affecting any of the results obtained in Sections 3 and 4).

E.6 Existence of our Equilibrium of Interest

We prove the existence of our equilibrium of interest under two parameter restrictions. The first restriction is

$$\psi < \bar{\psi}, \tag{E.13}$$

where $\bar{\psi}$ denotes the steady-state value of the reserves-to-loans ratio m_t/ℓ_t in our benchmark model (without liquid bonds). As we will see, this restriction will ensure that the reserve requirement (E.1) is not binding in our model with liquid bonds. The second restriction is

$$\max\left[1, \frac{1}{\beta} - \frac{z'(b^*)}{\beta\bar{\lambda}}\right] < I^m < \frac{1}{\beta}, \tag{E.14}$$

where $\bar{\lambda}$ denotes the upper bound of the steady-state values taken by the marginal utility of consumption λ_t as I^m varies from 1 to $1/\beta$ in our benchmark model (this upper bound being reached for $I^m = 1$). As we will see, that restriction will ensure that the interest rate on government bonds I_t^b is lower than the IOR rate I_t^m in our model with liquid bonds. In fact, that restriction will turn out to be sufficient but not necessary for $I_t^b < I_t^m$; for simplicity, we relegate to Appendix E.7 the statement of the (more complex) parameter restriction that is necessary and sufficient for $I_t^b < I_t^m$.

We proceed in two steps: we show first the existence of our *steady-state* equilibrium of interest, and then the existence of our *dynamic* equilibrium of interest. In the first step, to show that our model with liquid bonds, under the parameter restrictions (E.13) and (E.14), has a steady-state equilibrium with $I^b < I^m$ and $b^b = \lambda^r = 0$, we start from a candidate steady-state equilibrium with $b^b = \lambda^r = 0$. In this candidate equilibrium, as follows from the analysis above, all the endogenous variables that also appear in our benchmark model, except the lump-sum transfer T_t, take the same steady-state values as in that model. Using these values and $b^b = 0$, we then get residually the steady-state values of the other endogenous variables: (i) b^w and B from the market-clearing condition (E.8) and the steady-state target $B/P = b^*$; (ii) I^b from the first-order condition (E.9); and (iii) T from the consolidated budget constraint of the government (E.7).

At this stage, all equality conditions for steady-state equilibrium are satisfied, and the steady-state value of all endogenous variables is pinned down. What remains to be shown is that: (i) the inequality conditions for steady-state equilibrium, i.e. the steady-state versions of (E.1) and (E.12), are satisfied as strict inequalities, implying that the candidate steady-state equilibrium is indeed a steady-state equilibrium; and (ii) this equilibrium has the property that $I^b < I^m$. We first establish this last inequality by using in turn the first-order condition (E.9) with $I = 1/\beta$ and $b^w = b^*$, the inequality $\lambda \leq \bar{\lambda}$, and the parameter restriction (E.14), to get

$$I^b = \frac{1}{\beta} - \frac{z'(b^*)}{\beta\lambda} \leq \frac{1}{\beta} - \frac{z'(b^*)}{\beta\bar{\lambda}} < I^m.$$

In turn, the property $I^b < I^m$, together with $I^m < I$ and $\lambda^r = 0$, implies that the steady-state version of (E.12) is satisfied as a strict inequality:

$$\frac{I^b}{I} < (1-\eta) + \eta\frac{I^m}{I} + \eta\frac{\lambda^r}{\lambda}.$$

Finally, the parameter restriction (E.13) straightforwardly implies that the steady-state version of (E.1) is satisfied as a strict inequality. We conclude that our model with liquid bonds does indeed have a steady-state equilibrium with $I^b < I^m$ and $b^b = \lambda^r = 0$ that coincides with the steady-state equilibrium of our benchmark model. In this equilibrium, banks hold only reserves

($b^b = 0$) because they pay more interest than government bonds ($I^b < I^m$) and are at least as liquid as government bonds ($\eta \leq 1$).

In the second step, we proceed similarly to show the existence of a dynamic equilibrium with $I_t^b < I_t^m$ and $b_t^b = \lambda_t^r = 0$. More specifically, we start from a candidate equilibrium with $b_t^b = \lambda_t^r = 0$. In this candidate equilibrium, as follows from the analysis above, all the endogenous variables that also appear in our benchmark model, except the lump-sum transfer T_t, take the same equilibrium values as in that model. Using these values and $b_t^b = 0$, we then get residually the equilibrium values of the other endogenous variables (expressed as log-deviations from their steady-state values, and denoted by letters with hats): (i) $\hat{b}_t^w$ and $\hat{B}_t$ from the log-linearized version of the market-clearing condition (E.8) and the fiscal-policy rule; (ii) $\hat{I}_t^b$ from the log-linearized version of the first-order condition (E.9); and (iii) $\hat{T}_t$ from the log-linearized version of the government's consolidated budget constraint (E.7).

At this stage, all equality conditions for equilibrium are satisfied, and the equilibrium value of all endogenous variables is pinned down. What remains to be shown is that: (i) the inequality conditions for equilibrium, i.e. (E.1) and (E.12), are satisfied as strict inequalities, implying that the candidate equilibrium is indeed an equilibrium; and (ii) this equilibrium has the property that $I_t^b < I_t^m$. Now, we have just shown that these three strict inequalities are satisfied at the steady state; therefore, by continuity, they are also satisfied in the neighborhood of this steady state, under the standard assumption of small enough shocks. As a consequence, our model with liquid bonds does indeed have a dynamic equilibrium with $I_t^b < I_t^m$ and $b_t^b = \lambda_t^r = 0$, and this equilibrium coincides with the dynamic equilibrium of our benchmark model.

E.7 Relaxation of a Parameter Restriction

To prove the existence of our equilibrium of interest in Appendix E.6, we have used the parameter restriction (E.14), which involves the reduced-form parameter $\bar{\lambda}$. This restriction, however, can be harmlessly relaxed to some extent, because our proof only rests on the weaker condition

$$\max\left[1, \frac{1}{\beta} - \frac{z'(b^*)}{\beta\lambda}\right] < I^m < \frac{1}{\beta}, \tag{E.15}$$

where the steady-state value λ depends on several parameters of the model $-$ in particular β and I^m, but not b^*. To re-state (E.15) as a condition involving only parameters, we write λ as

$$\lambda = \Lambda\left(\beta I^m\right),$$

where the function Λ is defined by

$$\Lambda(x) \equiv u'\left\{f\left[\mathcal{F}^{-1}(x-1)\right]\right\}$$

for $x \in (-\infty, 1]$, where in turn the function $\mathcal{F}$ is defined in Subsection 3.1. Given the properties of $\mathcal{F}$, the function Λ is strictly decreasing ($\Lambda' < 0$), with $\lim_{x\to-\infty}\Lambda(x) = +\infty$. Therefore, there exists a unique $x^* \in (-\infty, 1)$ such that

$$1 - \frac{z'(b^*)}{\Lambda(x^*)} = x^*.$$

We can then re-state (E.15) as

$$\max\left(1, \frac{x^*}{\beta}\right) < I^m < \frac{1}{\beta},$$

where the reduced-form parameter x^* depends on several parameters of the model – in particular b^*, but not β nor I^m.

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