New Principles For Stabilization Policy

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Abstract: In a broad class of discrete-time rational-expectations models, I consider stabilization-policy rules making the policy instrument react with coefficient $\phi \in \mathbb{R}$ to a (past, current, or expected future) generic variable at time horizon $h \in \mathbb{Z}$, possibly among other variables. Using two complex-analysis theorems, I establish some simple, necessary or sufficient conditions on ϕ and h for these rules to ensure local-equilibrium determinacy. These conditions lead to new, general principles for stabilization policy. Building on these conditions, I characterize the circumstances under which (a generalized version of) the Taylor principle is necessary or sufficient for determinacy. I also provide the first hard guidelines for finding rules with robust determinacy properties across alternative models.

Keywords: stabilization policy, policy-instrument rule, local-equilibrium determinacy, Taylor principle, backward- and forward-looking policy, Rouché's theorem, Erdős-Turán theorem.

JEL codes: E32, E52.

1 Introduction

Dynamic rational-expectations models are widely used in macroeconomics. It is well known that these models can have "sunspot equilibria" in which the economy fluctuates around a steady state because of self-fulfilling expectations. Since these fluctuations are typically detrimental to welfare, a natural goal for stabilization policy is to eliminate these equilibria by ensuring "local-equilibrium determinacy" (i.e. existence and uniqueness of a stationary solution to the locally log-linearized model).

A large number of papers have thus studied, in various specific contexts, the conditions under which a policy-instrument rule ensures determinacy; that is, in discrete time, the inequality conditions on the coefficients of the rule for the resulting dynamic system to satisfy Blanchard and Kahn's (1980) determinacy conditions. Probably the best known result along these lines is about the so-called "Taylor principle" for monetary policy. Since Taylor (1993), monetary policy

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is commonly modeled by a simple interest-rate rule; in its simplest version, the Taylor principle states that the rule should make the interest rate react more than one-for-one to the inflation rate (when it reacts only to the inflation rate). This principle has been found to be necessary, and sometimes also sufficient, for determinacy in some simple prominent models for different inflation horizons in the rule (see, e.g., Woodford, 2003, Chapter 4).

Although some patterns emerge from this literature, no general result has yet been established. The Taylor principle is a good guide for determinacy in many monetary-policy models, but a poor one in others (see, e.g., Benhabib et al., 2001, Bilbiie, 2008). For monetary policy as for other stabilization policies, we lack a general understanding of determinacy outcomes depending on the model, the variables in the rule, and the coefficients and time horizons of these variables. We lack this understanding because the literature has derived determinacy conditions analytically only in simple models and for simple rules with short horizons. The main difficulty in getting more general analytical results is that Blanchard and Kahn's (1980) determinacy conditions are about the roots of the characteristic polynomial of the dynamic system; and these roots depend on (the coefficients and horizons of) the policy-instrument rule in a complicated way.

In this paper, I use two complex-analysis theorems to overcome this difficulty and establish some general, simple, necessary or sufficient conditions for determinacy in dynamic rational-expectations models. These conditions are directly about the coefficients and horizons of the policy-instrument rule, and lead to new principles for stabilization policy.

More specifically, I consider a general class of (locally log-linearized) discrete-time infinite-horizon rational-expectations models. Throughout most of the paper, I focus on (locally log-linearized) rules that make the policy instrument react to a single variable (or linear combination of variables) with coefficient $\phi \in \mathbb{R}$. The time horizon of this variable is $h \in \mathbb{Z}$: the policy instrument reacts to the $|h|$-period-lagged variable (when $h \leq -1$), the current variable (when $h = 0$), or the current expectation of the h-period-ahead variable (when $h \geq 1$). The determinacy status of the dynamic system composed of the model and the rule can be either "determinacy" (unique stationary solution), or "multiplicity" (infinity of stationary solutions), or "explosiveness" (no stationary solution). I characterize this determinacy status as a function of the coefficient ϕ and the horizon h in the rule, for $|\phi|$ sufficiently small or large, and/or for $|h|$ sufficiently large.

I distinguish between two cases, depending on whether a regularity condition is met or not. This regularity condition is only about the model and the variable in the rule; it does not involve the coefficient ϕ nor the horizon h of this variable.

When the regularity condition is met, there exists a positive threshold $\underline{\phi}$ such that for any $|\phi| < \underline{\phi}$, the determinacy status is independent of h and is the same as under a policy-instrument peg ($\phi = 0$). Intuitively, for $|\phi|$ sufficiently small, the rule does not change the system's dynamics enough to affect the determinacy status. Moreover, there exists a higher threshold $\bar{\phi}$ and a horizon $h^* \in \mathbb{Z}$ such that for any $|\phi| > \bar{\phi}$, there is explosiveness if $h \leq h^* - 1$, determinacy if

$h = h^*$, and multiplicity if $h \geq h^*+1$. Intuitively, for $|\phi|$ sufficiently large, the rule dominates the structural equations in the system's dynamics: a sufficiently large weight $|\phi|$ on outcomes before (resp. after) horizon h^* favors exploding (resp. imploding) paths and leads to explosiveness (resp. multiplicity).

For $|\phi| \in (\underline{\phi}, \bar{\phi})$, the structural equations do not entirely dominate the rule in the system's dynamics (since $|\phi| > \underline{\phi}$), nor does the rule entirely dominate the structural equations (since $|\phi| < \bar{\phi}$). As $|h| \to +\infty$, the roots of the system's characteristic polynomial distribute themselves between inside and outside the unit circle $\mathcal{C}$ of the complex plane in proportion of the share of $\mathcal{C}$ on which the structural equations dominate the rule and the share of $\mathcal{C}$ on which the rule dominates the structural equations. So, as $h \to -\infty$ (resp. as $h \to +\infty$), we eventually get more (resp. fewer) roots outside $\mathcal{C}$ than non-predetermined variables, and hence explosiveness (resp. multiplicity), for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$.

I address the question of whether the set of "determinacy horizons" (i.e. the set of horizons $h \in \mathbb{Z}$ such that determinacy obtains for at least one value of $\phi \in \mathbb{R}$) is bounded or not, below or above. This question matters for the desirability of backward- or forward-looking stabilization policy (and, in turn, backward-looking stabilization policy matters in the presence of "inside lags" – as are called recognition, decision, and implementation lags, which delay the reaction of policy to the state of the economy). The answer depends on whether the model delivers determinacy, multiplicity, or explosiveness under a policy-instrument peg ($\phi = 0$). In the monetary-policy literature, with the interest rate as the policy instrument, one comes across the three kinds of models, as I discuss below and as I illustrate in the main text.

For models that deliver determinacy under a peg, the set of determinacy horizons is unbounded below and above, since determinacy obtains for $|\phi| < \underline{\phi}$ at any horizon. For models that deliver multiplicity (resp. explosiveness) under a peg, the set of determinacy horizons is bounded above (resp. below). The reason is that large positive (resp. negative) horizons h do not much "perturb" the imploding (resp. exploding) equilibrium paths obtained under a peg, as the reaction of the policy instrument prescribed by the rule on these paths decreases exponentially with $|h|$; so, these horizons preserve the determinacy status obtained under a peg if this status is multiplicity (resp. explosiveness). For these models, the set of determinacy horizons can also be bounded both below and above; I establish sufficient conditions for this outcome to obtain.

I also study the validity of the Taylor principle as a necessary or sufficient condition for determinacy. I consider Woodford's (2001, 2003) version of the Taylor principle, sometimes called the long-run Taylor principle, which has a broader scope than the simpler version of the Taylor principle described above. I provide a formal, general definition of this principle, which applies to any stabilization-policy model and any variable in the rule. I characterize circumstances under which this principle is (alternatively) irrelevant, not necessary, not sufficient, sufficient, or locally necessary and sufficient for determinacy.

When the regularity condition mentioned above is not met, the results are partly modified. As I discuss and illustrate in the text, this condition is typically not met, for instance, when the variable in the rule is the first difference of a variable in the structural equations (e.g. the output growth rate in the rule and the output level in the structural equations), or when the variable in the rule is the output level and the long-run Phillips curve is vertical. In these cases, the results about the determinacy status for $|\phi| > \bar{\phi}$ are changed; those about the determinacy status for $|\phi| < \bar{\phi}$ and about the set of determinacy horizons are unchanged.

The regularity condition is also typically not met when, conversely, a variable in the structural equations is the first difference of the variable in the rule (e.g. the inflation rate in the structural equations and the price level in the rule). In that case, the results about the determinacy status for $|\phi|$ below or just above $\underline{\phi}$ are changed, and so are those about the set of determinacy horizons; the results about the determinacy status for $|\phi|$ above a neighborhood of $\underline{\phi}$ are unchanged.

I illustrate all these results, obtained under the regularity condition or not, with several simple monetary-policy examples. The models and calibrations are entirely borrowed from the literature. Even though the models are not quantitative, it is worth noting that $\underline{\phi}$ and $\bar{\phi}$ can be, in these examples, of the same order of magnitude as standard values of ϕ in the literature. Moreover, the highest determinacy horizon, when it exists, is not higher than 2 periods; the lowest one, when it exists, is not lower than -1 period (the period being typically a quarter).

Finally, I extend the results to rules involving several variables with different horizons and coefficients, one of which is a variable with horizon h and coefficient ϕ; and to inertial rules, i.e. rules involving the past values of the policy instrument in addition to a variable with horizon h and coefficient ϕ. The extended results show notably that for models delivering multiplicity under a policy-instrument peg, the set of determinacy horizons can always be made unbounded below and above with a "superinertial rule," i.e. a rule that would make the policy instrument explode over time if the variables set by the private sector were taken out of the rule.

A few remarks may serve to put my contribution in the context of the literature. The paper is, to my knowledge, the first to establish general determinacy conditions about the coefficients and horizons of policy-instrument rules. In particular, the concepts of $\underline{\phi}$, $\bar{\phi}$, and h^* are new. The literature has derived determinacy conditions analytically only in simple models and for simple rules with short horizons (so that the degree of the characteristic polynomial of the dynamic system is typically not higher than 3). Early examples of such contributions include Benhabib et al. (2001), Bullard and Mitra (2002), Carlstrom and Fuerst (2002), and Woodford (2003, Chapter 4), for horizons between -1 and 1. The two complex-analysis theorems that I use to establish my general results are those of Rouché (1862) and Erdős and Turán (1950). Bhattarai et al. (2014) use a stronger version of Rouché's theorem to derive a sufficient condition for determinacy in a model with partial price indexation and habit formation in consumption; this condition, unlike mine, is not directly about the coefficients and horizons of the rule.

Some of the results I establish are conditional on whether the model delivers multiplicity, determinacy, or explosiveness under a policy-instrument peg. One comes across the three types of models in the monetary-policy literature. Standard New Keynesian models typically deliver multiplicity under an interest-rate peg; this property is emphasized by Cochrane (2011, 2022); Giannoni and Woodford (2002) and Woodford (2003, Chapter 8) call it the "Sargent-Wallace property," after Sargent and Wallace (1975). Older models often deliver explosiveness under an interest-rate peg; this property is emphasized by Cochrane (2011, 2022), who calls these models "Old Keynesian." More recently, models have been developed that can deliver determinacy under an interest-rate peg (and, as a result, can solve some New Keynesian puzzles and paradoxes at the zero lower bound). Examples include the heterogenous-agents models of Acharya and Dogra (2020) and Bilbiie (2008, 2021), and the bounded-rationality model of Gabaix (2020).

The policy-instrument rules that I consider may involve any variable set by the private sector (at any positive or negative time horizon). In my numerical illustrations, I consider notably inflation, real output, nominal output, and the price level, all of which are classic variables in the monetary-policy literature; in particular, Woodford (2003) calls the interest-rate rules involving the price level "Wicksellian rules," after Wicksell (1898). Whether the variable(s) in the rule should be expressed in level or in growth rate is a long-standing issue in the literature (see, e.g., McCallum, 1999); my general results shed light on the implications of this choice for determinacy outcomes.

I also consider any type of inertial rules, including "first-difference rules," i.e. rules with a coefficient of unity on the lagged policy instrument (advocated in, e.g., Levin et al., 1999, and Levin and Williams, 2003), and "superinertial rules" (described above). As I discuss in the main text, my general results shed light notably on the determinacy implications of first-difference rules, on the poor performance of superinertial rules in Old Keynesian models (Rudebusch and Svensson, 1999, and Levin and Williams, 2003), and on the degree of superinertia of "robustly optimal rules" (Woodford, 2003, Chapter 8, and Giannoni and Woodford, 2002, 2003, 2005).

My results about positive determinacy horizons offer an explanation for the propensity of forward-looking interest-rate rules to generate multiplicity in New Keynesian models, as found in, e.g., Levin et al. (2003). Existing results on this front are mostly numerical and sparsely distributed across calibrated models and rules; my analytical results generalize them to a broad class of models and a broad class of rules (making the policy instrument react to any expected future variable). Woodford (1994) and Bernanke and Woodford (1997) were the first to warn against forward-looking rules on multiplicity grounds.

My results about negative determinacy horizons matter in the presence of inside lags. McCallum (1999) argues that rules need to take these lags into account to be operational. Benhabib (2004) analyzes the implications of inside lags for determinacy in a simple monetary-policy model (analytically in continuous time, numerically in discrete time); he argues that the lag structure may be long in discrete time – e.g., up to sixty periods if inflation changes twice daily and the interest rate is set according to inflation lagged thirty days. In Loisel (2021b), I investigate the ability of stabilization policy to ensure determinacy and uniquely implement a targeted equilibrium in the presence of inside or outside lags; the approach I take there (starting from a targeted characteristic polynomial and deriving a corresponding, arbitrarily complex policy-instrument rule) is radically different from the one I am taking here, and does not lead to any simple "principle" for stabilization policy.

Benhabib et al. (2001, 2003) derive determinacy conditions analytically for backward- and forward-looking interest-rate rules in a simple monetary-policy model, depending on whether prices are flexible or sticky and on how money enters preferences and technology. Their backward- and forward-looking rules differ from mine; in particular, their backward-looking rules amount to inertial rules, as Benhabib et al. (2003) note. Like Benhabib (2004), Benhabib et al. (2001, 2003) conduct most of their analysis in continuous time. The mathematical tools that I use are helpful for establishing general analytical determinacy conditions in discrete time, but not in continuous time.[1]

As I discuss in the main text, my general results provide guidelines for finding rules with robust determinacy properties across alternative models. The literature on the robustness of interest-rate rules across alternative monetary-policy models includes notably Levin et al. (1999, 2003), Levin and Williams (2003), Taylor and Williams (2011), and Wieland et al. (2012, 2016).

Most of the literature on policy-instrument rules is about monetary policy. The influential role of simple interest-rate rules in the actual conduct of monetary policy is documented in the contributions in Koenig et al. (2012), and discussed by the Federal Reserve (2018a, 2018b). But my results apply more generally to any stabilization policy. In particular, fiscal policy also raises indeterminacy issues, as first shown by Schmitt-Grohé and Uribe (1997).

I establish not only determinacy conditions, but also multiplicity conditions and explosiveness conditions. Clarida et al. (2000) and Lubik and Schorfheide (2004) have famously argued that US macroeconomic volatility before 1979 may be due to multiplicity. Beaudry et al. (2017, 2020) argue that recent US macroeconomic data are consistent with explosiveness (and convergence to a limit cycle).

Two limitations of my work are worth mentioning. First, like most of the related literature, I take the case of local-equilibrium multiplicity seriously. Some authors argue that an equilibrium-selection criterion should be used in this case, like the minimal-state-variable criterion of McCallum (1983) – for which Angeletos and Lian (2022) provide a recent formal justification – or the expectational-stability criterion of Evans (1985). From this alternative point of view, the distinction between determinacy and multiplicity may not matter anymore; but my results

[1] The determinacy status depends on the number of characteristic roots inside the unit circle of the complex plane (in discrete time), or on the number of characteristic roots in the left half-plane (in continuous time). Rouché's theorem, which I use, can directly characterize the former number, not the latter.

about explosiveness vs. either determinacy or multiplicity should still be of interest. Second, and again like most of the related literature (but unlike, e.g., Benhabib et al., 2001, 2002, 2003), I restrict attention to local equilibria, or their non-existence, in locally log-linearized models. This focus may not be that much of a limitation if non-local equilibria can be ruled out with the type of escape clause considered in, e.g., Benhabib et al. (2002); whether they can or cannot is, however, subject to debate (see, e.g., Cochrane, 2011, 2022).

The rest of the paper is organized as follows. Section 2 illustrates some of the main results of the paper in the basic New Keynesian model, with a rule making the interest rate react to inflation or output. The next two sections generalize the analysis to a broad class of models and to rules involving any single variable, depending on whether a regularity condition is met (Section 3) or not (Section 4). Section 5 extends the results to rules involving several variables and to inertial rules, and discusses some possible applications. I then conclude and provide a technical appendix.

2 A basic New Keynesian illustration

In this illustrative section, I derive some of the main results of the paper in a simple and well known context: the basic New Keynesian model, with a rule making the interest rate react to inflation or output. The analysis is a special case of the more general analysis conducted in the next section.

2.1 Determinacy status under Rule 1

I refer the reader to Woodford (2003) and Galí (2015) for a detailed presentation of the basic New Keynesian model. In this model, at each date $t \in \mathbb{Z}$, the private sector sets inflation π_t and output y_t according to the following (locally log-linearized) IS equation and Phillips curve:

$$y_t = \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\{\pi_{t+1}\}\right), \tag{1}$$

$$\pi_t = \beta\mathbb{E}_t\{\pi_{t+1}\} + \kappa y_t, \tag{2}$$

where $\mathbb{E}_t\{.\}$ denotes the date-t rational-expectations operator, and $\sigma > 0$, $\beta \in (0,1)$, and $\kappa > 0$ are three parameters.[2] I abstract from exogenous shocks in these structural equations, as they are irrelevant for determinacy issues. The policymaker is a central bank setting the short-term nominal interest rate i_t. I start with the case in which the central bank reacts to the past, current, or expected future inflation rate; i.e., I consider the following (locally log-linearized) interest-rate rule:

$$i_t = \phi\mathbb{E}_t\left\{\pi_{t+h}\right\}, \tag{Rule 1}$$

[2]I use the same notations for the parameters as in Galí (2015). Woodford (2003) uses the same notations for β and κ, but replaces $1/\sigma$ by σ.

where $(\phi, h) \in \mathbb{R} \times \mathbb{Z}$ (with $\mathbb{E}_t\{\pi_{t+h}\} = \pi_{t+h}$ when $h \leq 0$). I call ϕ and h the coefficient and horizon of inflation in the rule $-$ or, with slight abuse of language, the coefficient and horizon of the rule.

Using the Phillips curve (2) and Rule 1 to replace y_t, y_{t+1}, and i_t in the IS equation (1), I get the dynamic equation

$$\beta\mathbb{E}_t\{\pi_{t+2}\} - \left(1+\beta+\frac{\kappa}{\sigma}\right)\mathbb{E}_t\{\pi_{t+1}\} + \pi_t + \frac{\phi\kappa}{\sigma}\mathbb{E}_t\{\pi_{t+h}\} = 0.$$

Using the lag operator L, I rewrite this dynamic equation as

$$\mathbb{E}_t\{Q(L)\pi_{t+2}\} + \frac{\phi\kappa}{\sigma}\mathbb{E}_t\{\pi_{t+h}\} = 0, \tag{3}$$

where $Q(z) := \beta - (1+\beta+\kappa/\sigma)z + z^2 \in \mathbb{R}[z]$.[3] Let ν denote the number of non-predetermined variables of this dynamic equation, $P(z)$ the *reciprocal* polynomial of its characteristic polynomial, $\mathcal{C}$ the circle of radius 1 centered at the origin of the complex plane, and p the number of roots of $P(z)$ inside $\mathcal{C}$.[4] As follows from Blanchard and Kahn (1980), the dynamic equation has an infinity of stationary solutions if $p < \nu$, a unique stationary solution if $p = \nu$, and no stationary solution if $p > \nu$. I say that the "determinacy status" $S(\phi, h)$ of the system composed of the structural equations (1)-(2) and Rule 1 is equal to M (for "multiplicity") in the first case, D (for "determinacy") in the second case, and E (for "explosiveness") in the third case.

Under an interest-rate peg ($\phi = 0$), we have $\nu = 2$ (the two non-predetermined variables being $\mathbb{E}_t\{\pi_{t+1}\}$ and $\mathbb{E}_t\{\pi_{t+2}\}$) and $P(z) = Q(z)$. Since $Q(0) = \beta > 0$, $Q(1) = -\kappa/\sigma < 0$, and $\lim_{z\in\mathbb{R}, z\to+\infty} Q(z) = +\infty$, $Q(z)$ has one root in $(0,1)$ and another in $(1,+\infty)$. With $p = 1$ roots inside $\mathcal{C}$ for $\nu = 2$ non-predetermined variables, thus, the dynamic equation has an infinity of stationary solutions: $S(0, h) = M$ for any $h \in \mathbb{Z}$.

When the interest rate is not pegged ($\phi \neq 0$), we generically have $\nu = \max(2, h)$ (the non-predetermined variables being $\mathbb{E}_t\{\pi_{t+k}\}$ for $k \in \{1, \max(2,h)\}$) and

$$P(z) = Q(z)z^{\max(0,h-2)} + \frac{\phi\kappa}{\sigma}z^{\max(0,2-h)}.$$

This result is "generic" in the sense of holding for all $(\phi, h) \in (\mathbb{R}\setminus\{0\}) \times \mathbb{Z}$ except $(\phi, h) = (-\beta\sigma/\kappa, 2)$. If $(\phi, h) = (-\beta\sigma/\kappa, 2)$, then the coefficient of $\mathbb{E}_t\{\pi_{t+2}\}$ in the dynamic equation is 0, and we get $\nu = 1$ instead of $\nu = 2$. I study such zero-measure cases in detail in Loisel (2009); I ignore them in the present paper.

I determine the determinacy status $S(\phi, h)$ for $|\phi|$ sufficiently small or large, and/or for $|h|$ sufficiently large. I obtain the following results:[5]

[3] Throughout the paper, $\mathbb{R}[z]$ denotes the set of polynomials in z with real-number coefficients. Similarly, $\mathbb{C}[z]$ denotes the set of polynomials in z with complex coefficients.

[4] For any $\tilde{P}(z) \in \mathbb{R}[z]$ of degree d, the reciprocal polynomial of $\tilde{P}(z)$ is $z^d\tilde{P}(z^{-1})$. I work with the reciprocal polynomial of the characteristic polynomial, rather than with the characteristic polynomial itself, as the former is more convenient to use than the latter in conjunction with the lag operator.

[5] In this proposition and in the rest of the paper, I use the shortcut "$\forall\,|\phi|$ ..." for "$\forall\phi \in \mathbb{R}$ such that $|\phi|$ ...".

Proposition 1 (Determinacy status in the basic New Keynesian model under Rule 1): *Consider the basic New Keynesian model (1)-(2) with the rule* $i_t = \phi \mathbb{E}_t \{\pi_{t+h}\}$, *where* $(\phi, h) \in \mathbb{R} \times \mathbb{Z}$. *Let* $\underline{\phi} := (\sigma/\kappa) \min_{z \in \mathcal{C}} |Q(z)|$ *and* $\bar{\phi} := (\sigma/\kappa) \max_{z \in \mathcal{C}} |Q(z)|$. *Then:*

(a) $\forall |\phi| < \underline{\phi}$, $\forall h \in \mathbb{Z}$, $S(\phi, h) = M$;

(b) $\forall |\phi| > \bar{\phi}$, *(i)* $\forall h \leq -1$, $S(\phi, h) = E$, *(ii)* $S(\phi, 0) = D$, *(iii)* $\forall h \geq 1$, $S(\phi, h) = M$;

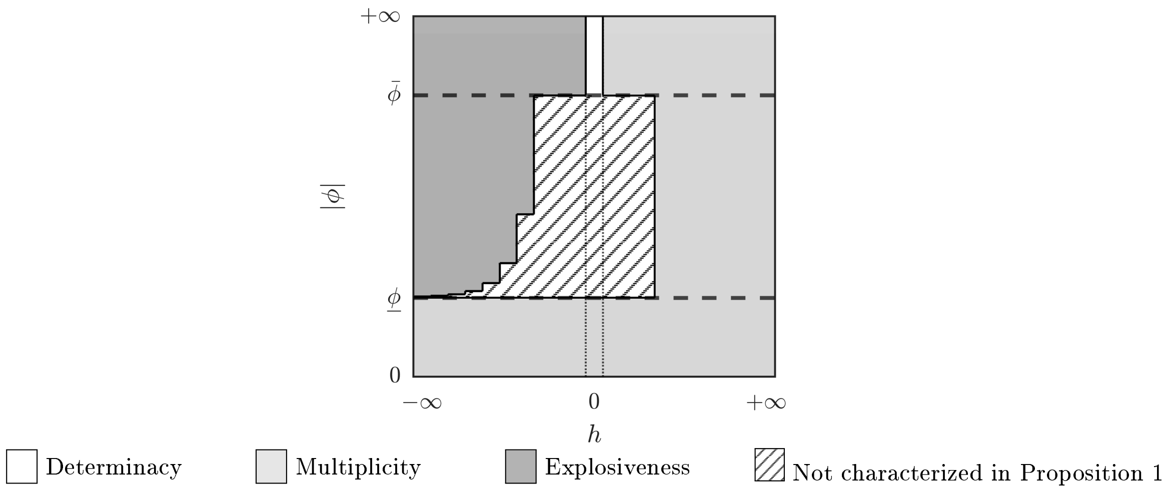
(c) $\exists \bar{h} \in \mathbb{Z}$, $\forall |\phi| \in (\underline{\phi}, \bar{\phi})$, $\forall h \geq \bar{h}$, $S(\phi, h) = M$;

(d) $\exists \underline{h} : (\underline{\phi}, \bar{\phi}) \to \mathbb{Z}$, *(i)* $\forall |\phi| \in (\underline{\phi}, \bar{\phi})$, $\forall h \leq \underline{h}(|\phi|)$, $S(\phi, h) = E$, *(ii)* $\forall \varepsilon \in (0, \bar{\phi} - \underline{\phi})$, $\underline{h}$ *is bounded on* $(\underline{\phi} + \varepsilon, \bar{\phi})$.

Proof: See Subsection 2.2 and Appendix A.1. ■

This proposition may look a bit cryptic at first sight. To enable the reader to grasp it at a glance, I represent it diagrammatically in Figure 1. This figure shows the determinacy status $S(\phi, h)$ in the pseudo half-plane $(h, |\phi|) \in \mathbb{Z} \times \mathbb{R}_+$, according to Proposition 1.

Figure 1: Determinacy status for the basic New Keynesian model and Rule 1



In the next two subsections, to convey the intuition behind Proposition 1, I prove Points (a)-(b) and I provide an outline of the proof of Points (c)-(d).

2.2 Proof of Points (a)-(b) of Proposition 1

Points (a)-(b) of Proposition 1 are about the determinacy status $S(\phi, h)$ for a sufficiently small or large absolute value of the coefficient ϕ. To prove these points, I use the theorem of Rouché (1862). I refer the reader to Henrici (1988, Theorem 4.10b, Page 280) for a general and modern statement of this theorem. Because I will apply it only to polynomials, I only need the following, more restrictive version of the theorem, where the term "Jordan curve" refers to a non-self-intersecting closed curve in the complex plane, and where the subscripts "b" and "s" stand respectively for "big" and "small":

Theorem 1 (Rouché, 1862): *Let $\mathcal{J}$ be a Jordan curve, $P_b(z) \in \mathbb{C}[z]$, and $P_s(z) \in \mathbb{C}[z]$. If $\forall z \in \mathcal{J}$, $|P_b(z)| > |P_s(z)|$, then $P_b(z) + P_s(z)$ and $P_b(z)$ have the same number of roots inside $\mathcal{J}$ (counting multiplicity).*

Proof: See Henrici (1988, Page 280). ■

To determine $S(\phi, h)$ for $|\phi|$ sufficiently small, I apply Rouché's theorem to $\mathcal{J} = \mathcal{C}$, $P_b(z) = Q(z)z^{\max(0,h-2)}$, and $P_s(z) = (\phi\kappa/\sigma)z^{\max(0,2-h)}$ (with, thus, $P_b(z) + P_s(z) = P(z)$). For any $|\phi| < \underline{\phi} := (\sigma/\kappa)\min_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})|$ and any $z \in \mathcal{C}$, we have

$$\left|Q(z)z^{\max(0,h-2)}\right| = |Q(z)| \geq \min_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})| = \frac{\underline{\phi}\kappa}{\sigma} > \frac{|\phi|\,\kappa}{\sigma} = \left|\frac{\phi\kappa}{\sigma}z^{\max(0,2-h)}\right|.$$

So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}$ as $Q(z)z^{\max(0,h-2)}$. The latter polynomial has exactly $\max(1, h-1)$ roots inside $\mathcal{C}$, since $Q(z)$ has exactly one root inside $\mathcal{C}$. Therefore, $p = \max(1, h-1) < \max(2, h) = \nu$, and we get $S(\phi, h) = M$ for any $h \in \mathbb{Z}$. Intuitively, for $|\phi|$ sufficiently small, the rule does not change the system's dynamics enough to affect the determinacy status, and this status remains the same as under an interest-rate peg (i.e., multiplicity).

To determine $S(\phi, h)$ for $|\phi|$ sufficiently large, I switch $P_b(z)$ and $P_s(z)$: i.e., I apply Rouché's theorem to $\mathcal{J} = \mathcal{C}$, $P_b(z) = (\phi\kappa/\sigma)z^{\max(0,2-h)}$, and $P_s(z) = Q(z)z^{\max(0,h-2)}$. For any $|\phi| > \bar{\phi} := (\sigma/\kappa)\max_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})|$ and any $z \in \mathcal{C}$, we have

$$\left|\frac{\phi\kappa}{\sigma}z^{\max(0,2-h)}\right| = \frac{|\phi|\,\kappa}{\sigma} > \frac{\bar{\phi}\kappa}{\sigma} = \max_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})| \geq |Q(z)| = \left|Q(z)z^{\max(0,h-2)}\right|.$$

So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}$ as $(\phi\kappa/\sigma)z^{\max(0,2-h)}$. The latter polynomial has exactly $\max(0, 2-h)$ roots inside $\mathcal{C}$; so, $p = \max(0, 2-h)$. Since $\nu = \max(2, h)$, we get: (i) if $h \leq -1$, then $p > \nu$ and $S(\phi, h) = E$; (ii) if $h = 0$, then $p = \nu$ and $S(\phi, h) = D$; and (iii) if $h \geq 1$, then $p < \nu$ and $S(\phi, h) = M$. Intuitively, for $|\phi|$ sufficiently large, the rule dominates the structural equations in the system's dynamics: a large weight $|\phi|$ on past inflation ($h \leq -1$) favors exploding paths and leads to explosiveness; a large weight $|\phi|$ on expected future inflation ($h \geq 1$) favors imploding paths and leads to multiplicity.

I show in Appendix A.2 that $\underline{\phi} = 1$ and $\bar{\phi} = 1 + 2(1+\beta)\sigma/\kappa$. That Rule 1 does not deliver determinacy in the basic New Keynesian model for $\phi \in (0, 1)$ and $\phi > 1 + 2(1+\beta)\sigma/\kappa$ is already known for $h = 1$ (see, e.g., Galí, 2015, Chapter 4, and Woodford, 2003, Chapter 4). Points (a)-(b) of Proposition 1 extend this result to any horizon $h \in \mathbb{Z} \setminus \{0\}$.

2.3 Outline of the proof of Points (c)-(d) of Proposition 1

Point (c) of Proposition 1 is about the existence of $\bar{h} \in \mathbb{Z}$ such that $S(\phi, h) = M$ for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \geq \bar{h}$. In my proof (in Appendix A.1), I do not seek to find the smallest integer $\bar{h}$ of that kind; I postpone this question to Subsection 2.6. Let z_o denote the root of $Q(z)$ in $(1, +\infty)$, with the subscript "o" standing for "outside $\mathcal{C}$." Consider a Jordan curve $\mathcal{J}_o$ surrounding z_o and not intersecting nor surrounding $\mathcal{C}$. I apply Rouché's theorem to $\mathcal{J} = \mathcal{J}_o$, $P_b(z) = Q(z)z^{h-2}$, and $P_s(z) = \phi\kappa/\sigma$. I obtain that for h sufficiently large, $P(z)$ has exactly one root inside $\mathcal{J}_o$, and hence at least one root outside $\mathcal{C}$, which implies $p < \nu$ and $S(\phi, h) = M$.

The intuition for this result is the following. Under an interest-rate peg ($\phi = 0$), we have a multiplicity of equilibrium paths that converge over time to zero at rate z_o^{-1}. When the interest rate is not pegged ($\phi \neq 0$), these paths are no longer equilibrium paths: they do not satisfy the dynamic equation (3) because of the term $(\phi\kappa/\sigma)\mathbb{E}_t\{\pi_{t+h}\}$ in this equation. When h is large, however, they are "close to satisfying" the dynamic equation, as the term $(\phi\kappa/\sigma)\mathbb{E}_t\{\pi_{t+h}\}$ is, on these paths, proportional to z_o^{-h} and hence close to zero. As a result, by continuity, there are neighboring paths that do satisfy the dynamic equation; i.e., there are equilibrium paths that converge over time to zero at a rate close to z_o^{-1}. As $h \to +\infty$, these equilibrium paths uniformly converge to those under an interest-rate peg, as the rate at which they converge over time to zero converges to z_o^{-1} (as can be readily checked by considering an arbitrarily small Jordan curve $\mathcal{J}_o$ around z_o in the reasoning above). In this sense, arbitrarily large horizons in the rule preserve all the local equilibria existing under an interest-rate peg.

Point (d) of Proposition 1 is about the determinacy status for $|\phi| \in (\underline{\phi}, \bar{\phi})$ and $-h$ sufficiently large. To prove this point in Appendix A.1, I consider a given $|\phi| \in (\underline{\phi}, \bar{\phi})$ and I proceed in four steps. In the first step, I show that all but one root of $P(z)$ converge uniformly to $\mathcal{C}$ as $h \to -\infty$. I get this result by applying Rouché's theorem twice: once to a circle approaching $\mathcal{C}$ from inside, and another time to a circle approaching $\mathcal{C}$ from outside. In the second step, I show that the roots of $P(z)$ uniformly converging to $\mathcal{C}$ as $h \to -\infty$ converge in distribution to the uniform distribution on $\mathcal{C}$. This result is a direct consequence of the second complex-analysis theorem that I use in the paper: the theorem of Erdős and Turán (1950), which I state in Appendix A.1.

In the third step, I consider an arc $\mathcal{A}$ of $\mathcal{C}$ on which the rule "dominates" the structural equations: $\forall z \in \mathcal{A}$, $|\phi| > |Q(z)|\,\sigma/\kappa \geq \underline{\phi}$. I use again Rouché's theorem to show that as $h \to -\infty$, any root of $P(z)$ close to $\mathcal{A}$ lies inside $\mathcal{C}$ – reflecting the fact that a sufficiently large weight on sufficiently ancient outcomes favors exploding paths. Given the result of the second step, therefore, the share of roots of $P(z)$ inside $\mathcal{C}$ is bounded below by $\ell(\mathcal{A})/\ell(\mathcal{C})$ as $h \to -\infty$, where $\ell(.)$ denotes the standard length operator (i.e., the Lebesgue measure on $\mathcal{C}$). As a result, as $h \to -\infty$, the number of roots of $P(z)$ inside $\mathcal{C}$ grows unboundedly ($p \to +\infty$) and eventually exceeds the constant number of non-predetermined variables ($\nu = 2$), leading to explosiveness. In the fourth step, finally, I use the fact that if $|\phi|$ is bounded away from $\underline{\phi}$, then $\ell(\mathcal{A})$ is bounded away from zero, and the function $\underline{h}(.)$ mentioned in Point (d) of Proposition 1 can be chosen bounded.

2.4 Determinacy status under Rule 2

I now replace inflation with output in the rule; i.e., I consider the rule

$$i_t = \phi \mathbb{E}_t \{y_{t+h}\}. \qquad \text{(Rule 2)}$$

Using the Phillips curve (2) and Rule 2 to replace y_t, y_{t+1}, and i_t in the IS equation (1), I get the following dynamic equation:

$$\beta \mathbb{E}_t \{\pi_{t+2}\} - \left(1 + \beta + \frac{\kappa}{\sigma}\right) \mathbb{E}_t \{\pi_{t+1}\} + \pi_t + \frac{\phi}{\sigma} \left(\mathbb{E}_t \{\pi_{t+h}\} - \beta \mathbb{E}_t \{\pi_{t+h+1}\}\right) = 0.$$

So, when the interest rate is not pegged ($\phi \neq 0$), there are now $\nu = \max(2, h+1)$ non-predetermined variables, and the reciprocal polynomial of the characteristic polynomial is

$$P(z) = Q(z) z^{\max(0,h-1)} + \frac{\phi}{\sigma} (z - \beta) z^{\max(0,1-h)}.$$

It is easy to conduct the same analysis as in the previous subsections, replacing $Q(z)z^{\max(0,h-2)}$, $(\phi\kappa/\sigma)z^{\max(0,2-h)}$, $\underline{\phi} := (\sigma/\kappa) \min_{z \in \mathcal{C}} |Q(z)|$, and $\bar{\phi} := (\sigma/\kappa) \max_{z \in \mathcal{C}} |Q(z)|$ by, respectively, $Q(z)z^{\max(0,h-1)}$, $(\phi/\sigma)(z-\beta)z^{\max(0,1-h)}$, $\underline{\phi} := \sigma \min_{z \in \mathcal{C}} |Q(z)/(z-\beta)|$, and $\bar{\phi} := \sigma \max_{z \in \mathcal{C}} |Q(z) /(z-\beta)|$. Since $\beta \in (0,1)$, the results are unchanged: Points (a)-(d) of Proposition 1 still hold with the new thresholds $\underline{\phi}$ and $\bar{\phi}$.

Proposition 2 (Determinacy status in the basic New Keynesian model under Rule 2): *Consider the basic New Keynesian model (1)-(2) with the rule $i_t = \phi \mathbb{E}_t \{y_{t+h}\}$, where $(\phi, h) \in \mathbb{R} \times \mathbb{Z}$. Let $\underline{\phi} := \sigma \min_{z \in \mathcal{C}} |Q(z)/(z-\beta)|$ and $\bar{\phi} := \sigma \max_{z \in \mathcal{C}} |Q(z)/(z-\beta)|$. Then, Points (a)-(d) of Proposition 1 still hold.*

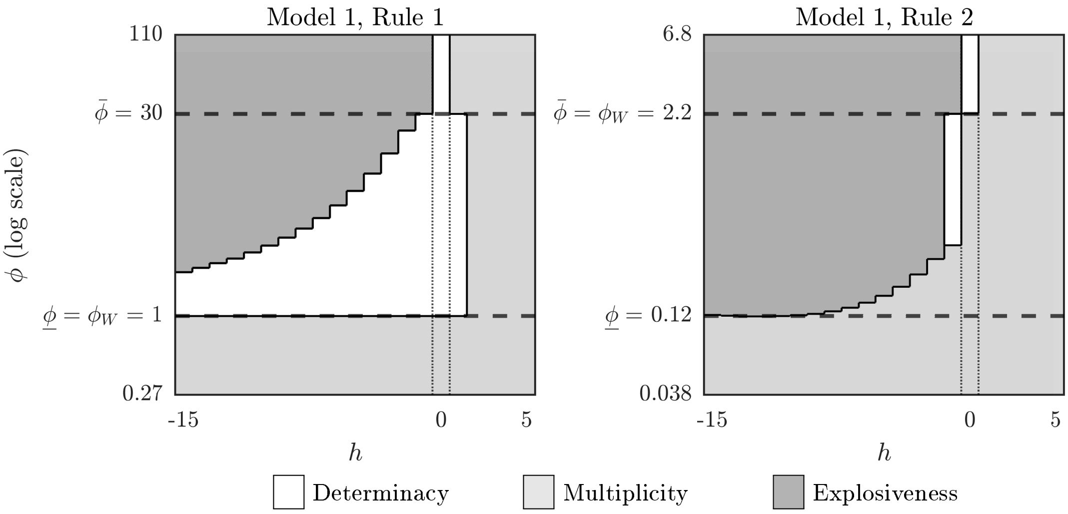
Proposition 2 can be represented in exactly the same diagrammatic form as Proposition 1: Figure 1 shows the determinacy status in the basic New Keynesian model not only under Rule 1, but also under Rule 2. The analytical expression of the thresholds $\underline{\phi}$ and $\bar{\phi}$ under Rule 2 are determined in Appendix A.3.

2.5 Numerical example

In order to illustrate Proposition 1-2 and the next propositions numerically, I consider Woodford's (2003, Chapter 4) calibration of the basic New Keynesian model: $(\beta, \kappa, \sigma) = (0.99, 0.022, 0.16)$, the period being one quarter. I call "Model 1" the resulting calibrated model, as it is the first of several calibrated models that I will consider in the paper.

The results obtained for Model 1 and Rules 1-2 are presented in Figure 2. This figure represents the determinacy status $S(\phi, h)$ in the pseudo half-plane $(h, \phi) \in \mathbb{Z} \times \mathbb{R}_+$ with a log scale for ϕ. I focus on positive values of ϕ for consistency with the theoretical and empirical literatures. The

Figure 2: Determinacy status for Model 1 and Rules 1-2



coefficient threshold ϕ_W featuring in the figure will be introduced and commented upon in the next subsection.

Even though the basic New Keynesian model is clearly not a quantitative model, a few features of this figure are worth emphasizing. First, the horizon threshold at and above which the rule can no longer deliver determinacy (i.e., the lowest integer $\bar{h}$ in Point (c) of Propositions 1-2) is two quarters for Rule 1, and one quarter for Rule 2. So, in this numerical example, monetary policy should hardly be forward-looking (under Rule 1), or not be forward-looking at all (under Rule 2), in order to ensure determinacy. In the next subsection, I will determine analytically this horizon threshold for Rule 1, and I will argue that it is typically low.

Second, Rule 2 can no longer deliver determinacy for a horizon equal to or lower than minus two quarters. So, in this numerical example, a central bank that would react to output with a delay of two or more quarters, say because of data-publication lags, would necessarily be behind the curve and fail to ensure determinacy, no matter how strongly or weakly it reacts to output. In Subsection 2.7, I will derive analytically a necessary and sufficient condition for existence of such a horizon threshold at and below which Rule 2 can no longer deliver determinacy.

Third, compared to standard values of ϕ in the literature (often between 0.5 and 2), the lower coefficient threshold $\underline{\phi}$ is of the same order of magnitude or one order of magnitude smaller, while the upper coefficient threshold $\bar{\phi}$ is of the same order of magnitude or one order of magnitude larger.

In addition, I have also considered Galí's (2015, Chapter 3) calibration of the basic New Keynesian model: $(\beta, \kappa, \sigma) = (0.99, 0.125, 1)$, the period being again one quarter. Most of the results under this alternative calibration are qualitatively and quantitatively similar. In particular, determinacy can again be obtained only for $h \leq 1$ under Rule 1, and only for $h \in \{-1, 0\}$ under Rule 2. The only notable difference is that $\underline{\phi}$ and $\bar{\phi}$ for Rule 2 are roughly multiplied by a factor of 6 (essentially because of the difference in the value of σ between the two calibrations).

2.6 Determinacy horizons and Taylor principle under Rule 1

I now characterize more precisely the set of horizons for which Rules 1 and 2 can deliver determinacy, i.e. the set $H_D := \{h \in \mathbb{Z} | \exists \phi \in \mathbb{R}, S(\phi, h) = D\}$ (which I call the set of "determinacy horizons"), in order to assess the desirability of forward- and backward-looking monetary policy. I also examine the validity of the Taylor principle as a condition for determinacy. I do that for Rule 1 in this subsection, and for Rule 2 in the next subsection.

Before stating the results, I need to specify what I mean by "Taylor principle." There are several versions of this principle in the literature. The simplest and narrowest version is that the rule should make the interest rate react more than one-for-one to the inflation rate, when it reacts only to the inflation rate. Another, more general version, sometimes called the long-run Taylor principle, was proposed by Woodford (2001, 2003) and is used in, e.g., Galí (2015, Chapter 4). For the sake of generality, I adopt the latter version of the Taylor principle − which, in the basic New Keynesian model under Rule 1, amounts anyway to the former version.

I will provide a formal, general definition of the long-run Taylor principle in Section 3. In the current section, I only need to state this principle in the specific context of the basic New Keynesian model, under Rule 1 or 2. In this context, loosely speaking, the long-run Taylor principle states that if the inflation rate were permanently higher by one percentage point, then the system composed of the Phillips curve (2) and Rule 1 or 2 should make the interest rate permanently higher by more than one percentage point. Under Rule 1, this principle straightforwardly translates into $\phi > \phi_W := 1$, where the subscript W stands for "Woodford." Under Rule 2, this principle amounts to $\phi > \phi_W := \kappa/(1-\beta)$, since the Phillips curve (2) implies that a permanent increase in inflation of one percentage point leads to a permanent increase in output of $(1-\beta)/\kappa$ percentage points.

I can now state the results for Rule 1 as follows:

Proposition 3 (Determinacy horizons and Taylor principle in the basic New Keynesian model under Rule 1): *Consider the basic New Keynesian model (1)-(2) with the rule* $i_t = \phi \mathbb{E}_t \{\pi_{t+h}\}$, *where* $(\phi, h) \in \mathbb{R} \times \mathbb{Z}$. *Then* $\phi_W = \underline{\phi}$ *and:*
(a) $H_D = \{h \in \mathbb{Z} | h < 1 + (1-\beta)\sigma/\kappa\}$;
(b) $\forall h \in \mathbb{Z} \setminus H_D$, *the Taylor principle is irrelevant for* D;
(c) $\forall h \in H_D$, *the Taylor principle is locally necessary and sufficient for* D;
(d) $\forall h \in H_D$, *the Taylor principle is sufficient for* D *if and only if* $h = 0$.

Proof: See Appendix A.4. ■

Point (a) of this proposition characterizes the set H_D. It gives the analytical expression of the smallest integer $\bar{h}$ in Point (c) of Proposition 1. In standard calibrations of the basic New Keynesian model, the horizon threshold $1 + (1 - \beta)\sigma/\kappa$ is typically between 1 and 2, because β is typically set to 0.99 (on a quarterly basis). In the two calibrations considered in the previous subsection, in particular, this threshold takes the values 1.07 and 1.08, which are much closer to 1 than to 2. So, in the basic New Keynesian model, a central bank reacting to inflation should hardly be forward-looking, if at all, in order to ensure determinacy.

Point (a) of Proposition 3 also says that a central bank reacting to inflation can be arbitrarily backward-looking and still ensure determinacy; but, as we know from Point (d) of Proposition 1, the set of values for ϕ leading to determinacy gradually shrinks to the empty set as $h \to -\infty$.

Points (b)-(d) of Proposition 3 are about the Taylor principle as a condition for determinacy. Point (b) straightforwardly follows from the definition of H_D. Point (d) of Proposition 3 essentially follows from Point (b) of Proposition 1. Point (c) states a local result, i.e. a result holding for ϕ in the neighborhood of $\phi_W = 1$. As ϕ crosses ϕ_W from below, one root of $P(z)$ crosses $\mathcal{C}$ at point 1. When $h < 1+(1-\beta)\sigma/\kappa$, the root goes from outside to inside $\mathcal{C}$, so the determinacy status moves from multiplicity to determinacy. Alternatively, when $h > 1+(1-\beta)\sigma/\kappa$, the root goes from inside to outside $\mathcal{C}$, so the determinacy status remains multiplicity. All these results about the Taylor principle can be seen in the left panel of Figure 2, where I have featured ϕ_W.

2.7 Determinacy horizons and Taylor principle under Rule 2

I now turn to Rule 2: I characterize again, this time partially, the set of determinacy horizons H_D, and I study again the validity of the Taylor principle as a condition for determinacy. As discussed in the previous subsection, the Taylor principle under Rule 2 is $\phi > \phi_W := \kappa/(1-\beta)$. Let me define

$$\eta := \left[(1-\beta)^2 + \left(1+\beta+\frac{\kappa}{\sigma}\right)^2\right]\beta - (1+\beta)\left(1+\beta^2\right)\left(1+\beta+\frac{\kappa}{\sigma}\right).$$

I can then state the results for Rule 2 as follows:

Proposition 4 (Determinacy horizons and Taylor principle in the basic New Keynesian model under Rule 2): *Consider the basic New Keynesian model (1)-(2) with the rule* $i_t = \phi\mathbb{E}_t\{y_{t+h}\}$, *where* $(\phi, h) \in \mathbb{R}\times\mathbb{Z}$. *Then:*

(a) H_D *is bounded above;*

(b) H_D *is bounded below if and only if* $\left|\eta - 4\beta^2\right| < 4\beta\left(1+\beta^2\right)$*;*

(c) $\forall h \in \mathbb{Z}\setminus H_D$, *the Taylor principle is irrelevant for D;*

(d) $\forall h \in H_D$, *the Taylor principle is sufficient for D if and only if* $h = 0$*;*

(e) if (and only if) $\kappa/\sigma \geq (1+\beta)(1-\beta)^2/\beta$, *then: (i)* $\phi_W = \bar{\phi}$, *(ii)* $\forall h \leq -1$, *the Taylor principle is sufficient for E, (iii) for* $h = 0$, *it is sufficient for D, (iv)* $\forall h \geq 1$, *it is suff. for M;*

(f) if (and only if) $\eta - 4\beta^2 < -4\beta\left(1+\beta^2\right)$, *then: (i)* $\phi_W = \underline{\phi}$, *(ii)* $\forall h \in \mathbb{Z}$, *the Taylor principle is locally necessary and sufficient for D if and only if* $h < (1-\beta)(1+\sigma/\kappa)$.

Proof: see Appendix A.5. ■

Points (a) and (c) of this proposition straightforwardly follow from, respectively, Point (c) of Proposition 2 and the definition of H_D. Points (d)-(e) of Proposition 4 essentially follow from Point (b) of Proposition 2. Note that Point (e) of Proposition 4 has no counterpart in Proposition 3; the reason is that under Rule 1, unlike under Rule 2, we necessarily have $\phi_W = \underline{\phi}$ and hence $\phi_W \neq \bar{\phi}$ (as stated in Proposition 3).

Point (b) of Proposition 4 states the necessary and sufficient condition on the structural parameters for H_D to be bounded below. If this condition is met, then a central bank reacting to output with a sufficiently long delay will necessarily generate either multiplicity or explosiveness, no matter how strongly or weakly it reacts to output. The condition stated in this point is, in fact, necessary and sufficient for $\arg\min_{z\in\mathcal{C}} |Q(z)/(z-\beta)| \subset \mathcal{C} \setminus \{-1, 1\}$, i.e. for the minimum in the definition of $\underline{\phi}$ to be obtained for some non-real z's, whose number is necessarily even. So, as $|\phi|$ crosses $\underline{\phi}$ from below, an even number of roots of $P(z)$ cross $\mathcal{C}$, either from inside to outside, or vice-versa. To move from multiplicity to determinacy, however, we would need exactly one root of $P(z)$ to go from outside to inside $\mathcal{C}$. So, the determinacy status either remains multiplicity, or jumps directly from multiplicity to explosiveness. For $-h$ sufficiently large, if no $|\phi|$ in the neighborhood of $\underline{\phi}$ can ensure determinacy, then more generally no $|\phi| \in \mathbb{R}_+$ can ensure determinacy. Note that Point (b) of Proposition 4 has no counterpart in Proposition 3; the reason is that under Rule 1, unlike under Rule 2, the minimum in the definition of $\underline{\phi}$ is necessarily obtained for $z = 1$: $\arg\min_{z\in\mathcal{C}} |Q(z)| = \{1\} \not\subset \mathcal{C} \setminus \{-1, 1\}$.

The conditions stated in Point (b) and (e) of Proposition 4 are met in Model 1, as apparent in the right panel of Figure 2 (where I have featured ϕ_W). They are also met under the other calibration considered in Subsection 2.5.

Point (f) of Proposition 4 is similar to Point (c) of Proposition 3. Under the condition stated in this point, one root of $P(z)$ crosses $\mathcal{C}$ at point 1 as ϕ crosses ϕ_W from below. When $h < (1-\beta)(1+\sigma/\kappa)$, the root goes from outside to inside $\mathcal{C}$, so the determinacy status moves from multiplicity to determinacy. Alternatively, when $h > (1-\beta)(1+\sigma/\kappa)$, the root goes from inside to outside $\mathcal{C}$, so the determinacy status remains multiplicity.

3 Determinacy analysis for regular systems

In this section, I generalize the results of the previous section to a broad class of dynamic rational-expectations models, and to rules making the policy instrument react to any variable (or linear combination of variables) at horizon h with coefficient ϕ. I focus on models and rules that make the dynamic system "regular" in a sense that I specify below; I postpone the analysis of non-regular systems to the next section.

3.1 Model and rule

At each date $t \in \mathbb{Z}$, the private sector sets an n-dimension vector of endogenous variables $\mathbf{X}_t$ according to the following (locally log-linearized) structural equations:

$$\mathbb{E}_t \left\{ \mathbf{\Delta} \left(L^{-1} \right) \left[\mathbf{A} \left(L \right) \mathbf{X}_t + L^{-\gamma} \mathbf{B} \left(L \right) i_t \right] \right\} = \mathbf{0}, \tag{4}$$

where again i_t denotes the policy instrument at date t, L the lag operator, and $\mathbb{E}_t\{.\}$ the date-t rational-expectations operator. I abstract again from exogenous shocks, as they are irrelevant for determinacy issues. These structural equations are parameterized by $n \in \mathbb{N} \setminus \{0\}$, $\gamma \in \mathbb{N}$, $\mathbf{A}(z) \in \mathbb{R}^{n \times n}[z]$, $\mathbf{B}(z) \in \mathbb{R}^{n \times 1}[z]$, and $\mathbf{\Delta}(z) \in \mathbb{R}^{n \times n}[z]$ which is a diagonal matrix whose j^{th} diagonal element is z^{δ_j} with $\delta_j \in \mathbb{N}$.[6]

I make two non-restrictive assumptions on $\mathbf{A}(z)$ and $\mathbf{B}(z)$. First, I assume that $\det[\mathbf{A}(0)] \neq 0$; this assumption is made without any loss in generality because any system of independent structural equations of type (4) that does not satisfy this assumption can be equivalently rewritten as a system of type (4) that does. Second, I assume that $\mathbf{B}(z) \neq \mathbf{0}$; this assumption is needed simply for the policy instrument to have some effect on the endogenous variables set by the private sector.

The structural equations (1)-(2) of the basic New Keynesian model, in particular, can straightforwardly be written in a form of type (4) with $n = 2$, $\gamma = 0$,

$$\mathbf{X}_t = \begin{bmatrix} y_t \\ \pi_t \end{bmatrix}, \ \mathbf{\Delta}(z) = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}, \ \mathbf{A}(z) = \begin{bmatrix} 1 - z & \frac{1}{\sigma} \\ \kappa z & \beta - z \end{bmatrix}, \text{ and } \mathbf{B}(z) = \begin{bmatrix} -\frac{z}{\sigma} \\ 0 \end{bmatrix}.$$

This system satisfies the two assumptions made above: $\det[\mathbf{A}(0)] = \beta \neq 0$, and $\mathbf{B}(z) \neq \mathbf{0}$.

The policymaker follows the following (log-linearized) policy-instrument rule:

$$i_t = \phi \mathbb{E}_t \left\{ v_{t+h} \right\}, \tag{5}$$

where again $\phi \in \mathbb{R}$ and $h \in \mathbb{Z}$ (with $\mathbb{E}_t \left\{ v_{t+h} \right\} = v_{t+h}$ when $h \leq 0$), and where v_t can be any linear combination of current and past endogenous variables: $v_t := \mathbf{V}(L)\mathbf{X}_t$, with $\mathbf{V}(z) \in \mathbb{R}^{1 \times n}[z]$. I make the following non-restrictive assumption on $\mathbf{V}(z)$:

$$W(z) := \det \begin{bmatrix} \mathbf{A}(z) & \mathbf{B}(z) \\ \mathbf{V}(z) & 0 \end{bmatrix} \neq 0.$$

If this assumption were not satisfied, then v_t could be expressed as a linear combination of (a backward-looking version of) the structural equations, and would therefore be exogenous. In the basic New Keynesian model, for instance, we have $W(z) = [(\beta - z)V_1(z) - \kappa z V_2(z)]z/\sigma$, where $V_1(z)$ and $V_2(z)$ denote the two elements of $\mathbf{V}(z)$; so, imposing $W(z) \neq 0$ amounts to

[6] Throughout the paper, letters in bold denote vectors and matrices that have potentially more than one element. $\mathbf{0}$ denotes a vector or a matrix whose elements are all equal to zero and whose dimensions depend on the specific context in which it is used. For any $(n_1, n_2) \in (\mathbb{N} \setminus \{0\})^2$, $\mathbb{R}^{n_1 \times n_2}[z]$ denotes the set of polynomials in z whose coefficients are $n_1 \times n_2$ matrices with real-number elements.

ruling out variables v_t of type $v_t = \tilde{V}(L)(\pi_{t-1} - \beta\pi_t - \kappa y_{t-1})$ with $\tilde{V}(z) \in \mathbb{R}[z]$, i.e. variables v_t that are "proportional" to a backward-looking version of the Phillips curve (2). Such variables are exogenous because they can be rewritten, using the Phillips curve (2), as the sum of past expectation errors: $v_t = -\beta\tilde{V}(L)(\pi_t - \mathbb{E}_{t-1}\{\pi_t\})$. For a horizon h higher than the degree of $\tilde{V}(z)$, in particular, the term $\mathbb{E}_t\{v_{t+h}\}$ in the rule (5) is simply zero.

3.2 Determinacy status

As in Section 2, let ν denote the number of non-predetermined variables of the system (4)-(5), and $P(z)$ the reciprocal polynomial of the characteristic polynomial of this system. In addition, let $\omega \in \mathbb{N}$ denote the multiplicity of 0 as a root of $W(z)$ (with $\omega = 0$ if $W(0) \neq 0$). I start by establishing a useful preliminary result:

Lemma 1: *If $\phi = 0$, then $\nu = \delta := \sum_{j=1}^{n} \delta_j$ and $P(z) = Q(z) := \det[\mathbf{A}(z)]$. If $\phi \neq 0$, then, except possibly for a zero-measure set of values of ϕ, $\nu = \delta + \max(0, h - m)$ and*

$$P(z) = Q(z)z^{\max(0,h-m)} + \phi R(z)z^{\max(0,m-h)}, \tag{6}$$

where $m := \omega - \gamma$ and $R(z) := -z^{-\omega}W(z)$.

Proof: See Appendix A.6. ■

This lemma generalizes similar preliminary results obtained in Section 2: in the specific context of the basic New Keynesian model, we had $Q(z) = \beta - (1 + \beta + \kappa/\sigma)z + z^2$ and $\delta = 2$, with $m = 2$ and $R(z) = \kappa/\sigma$ for Rule 1, and $m = 1$ and $R(z) = (z - \beta)/\sigma$ for Rule 2. The "zero-measure set of values of ϕ" mentioned in the lemma refers again to the possibility of reducing ν below $\delta + \max(0, h - m)$ with carefully designed policy-instrument rules as in Loisel (2009), a possibility that I ignore in the present paper.

The polynomial $Q(z)$ depends on the model (4), not on the rule (5). It is the reciprocal polynomial of the characteristic polynomial under a policy-instrument peg, i.e. under the policy-instrument rule $i_t = 0$. The polynomial $R(z)$ depends on the model (4) and the variable v_t in the rule (5), not on the coefficient ϕ nor on the horizon h of the rule (5). It is the reciprocal polynomial of the characteristic polynomial under the "targeting rule" $v_t = 0$.

Let $q_{\mathcal{C}} := \#\{z \in \mathcal{C}|Q(z) = 0\}$ and $r_{\mathcal{C}} := \#\{z \in \mathcal{C}|R(z) = 0\}$ denote the number of roots of $Q(z)$ and $R(z)$ on $\mathcal{C}$ (counting multiplicity). I distinguish between two cases : the "regular case" in which $q_{\mathcal{C}} = r_{\mathcal{C}} = 0$ (as in Section 2), and the "non-regular case" in which $q_{\mathcal{C}} \geq 1$ or $r_{\mathcal{C}} \geq 1$. I focus on the regular case in the current section, and I will address the non-regular case in the next section.

I also distinguish between three kinds of models, depending on their determinacy status under a policy-instrument peg ($\phi = 0$). Let $p := \#\{z \in \mathbb{C}|P(z) = 0, |z| < 1\}$ and $q := \#\{z \in \mathbb{C}|Q(z) =$

$0, |z| < 1\}$ denote the number of roots of $P(z)$ and $Q(z)$ inside $\mathcal{C}$ (counting multiplicity). As follows from Lemma 1, $p = q$ and $\nu = \delta$ under a policy-instrument peg. As a result, Blanchard and Kahn's (1980) *root-counting* condition for determinacy, $p = \nu$, is met under a peg if and only if $q = \delta$. I assume here, and everywhere else in the paper, that Blanchard and Kahn's (1980) *no-decoupling* condition is met (it is straightforward to check that it is met in all the specific examples I consider in the paper).[7] As a result, the determinacy status under a peg is multiplicity for models with $q \leq \delta - 1$ ($\forall h \in \mathbb{Z}$, $S(0, h) = M$), determinacy for models with $q = \delta$ ($\forall h \in \mathbb{Z}$, $S(0, h) = D$), and explosiveness for models with $q \geq \delta + 1$ ($\forall h \in \mathbb{Z}$, $S(0, h) = E$). As I document in the Introduction, one comes across the three types of models in the monetary-policy literature.

Using Lemma 1, it is easy to conduct the same analysis as in Section 2, and thus to generalize Propositions 1-2 to the class of models (4) and the class of rules (5). I obtain the following proposition, where $r := \#\{z \in \mathbb{C} | R(z) = 0, |z| < 1\}$ denotes the number of roots of $R(z)$ inside $\mathcal{C}$ (counting multiplicity):

Proposition 5 (Determinacy status for systems with $q_{\mathcal{C}} = r_{\mathcal{C}} = 0$): *Consider a model (4) and a variable v_t such that $q_{\mathcal{C}} = r_{\mathcal{C}} = 0$. Let $\underline{\phi} := \min_{z \in \mathcal{C}} |Q(z)/R(z)|$, $\bar{\phi} := \max_{z \in \mathcal{C}} |Q(z)/R(z)|$, and $h^* := m + r - \delta$. Then, under the rule $i_t = \phi \mathbb{E}_t \{v_{t+h}\}$ with $(\phi, h) \in \mathbb{R} \times \mathbb{Z}$:*

(a) $\forall |\phi| < \underline{\phi}$, $\forall h \in \mathbb{Z}$, (i) if $q \leq \delta - 1$, then $S(\phi, h) = M$, (ii) if $q = \delta$, then $S(\phi, h) = D$, (iii) if $q \geq \delta + 1$, then $S(\phi, h) = E$;

(b) $\forall |\phi| > \bar{\phi}$, (i) $\forall h \leq h^* - 1$, $S(\phi, h) = E$, (ii) $S(\phi, h^*) = D$, (iii) $\forall h \geq h^* + 1$, $S(\phi, h) = M$;

(c) $\exists \bar{h} : (\underline{\phi}, \bar{\phi}) \to \mathbb{Z}$, (i) $\forall |\phi| \in (\underline{\phi}, \bar{\phi})$, $\forall h \geq \bar{h}(|\phi|)$, $S(\phi, h) = M$, (ii) $\forall \varepsilon \in (0, \bar{\phi} - \underline{\phi})$, $\bar{h}$ is bounded on $(\underline{\phi} + \varepsilon, \bar{\phi})$, (iii) if $q \leq \delta - 1$, then $\bar{h}$ is bounded on $(\underline{\phi}, \bar{\phi})$;

(d) $\exists \underline{h} : (\underline{\phi}, \bar{\phi}) \to \mathbb{Z}$, (i) $\forall |\phi| \in (\underline{\phi}, \bar{\phi})$, $\forall h \leq \underline{h}(|\phi|)$, $S(\phi, h) = E$, (ii) $\forall \varepsilon \in (0, \bar{\phi} - \underline{\phi})$, $\underline{h}$ is bounded on $(\underline{\phi} + \varepsilon, \bar{\phi})$, (iii) if $q \geq \delta + 1$, then $\underline{h}$ is bounded on $(\underline{\phi}, \bar{\phi})$.

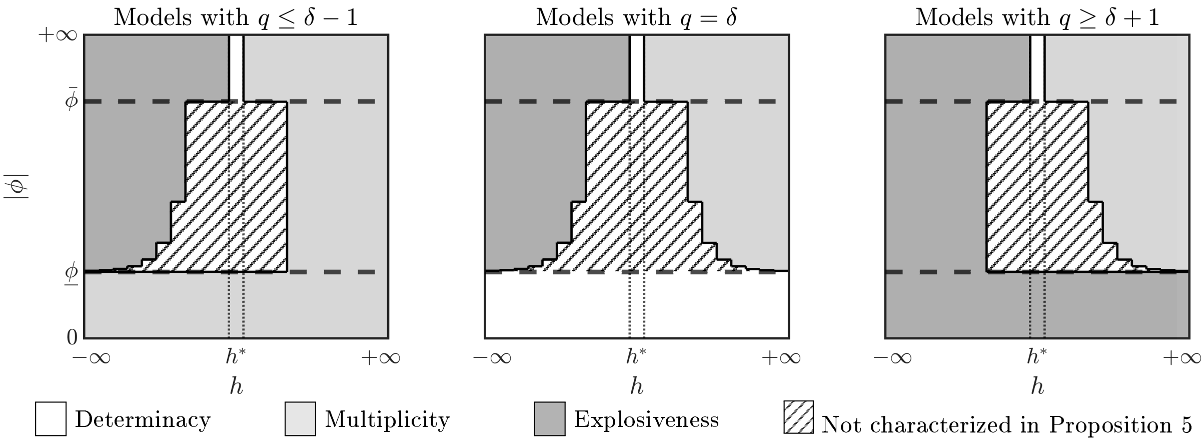
Proof: See Appendix A.7. ■

Like Propositions 1-2, this proposition may look a bit cryptic at first sight. Like Propositions 1-2, however, it can be represented in a simple diagrammatic form: Figure 3 shows the determinacy status $S(\phi, h)$ in the pseudo half-plane $(h, |\phi|) \in \mathbb{Z} \times \mathbb{R}_+$, according to Proposition 5.

The intuitions behind Proposition 5 are identical or similar to those behind Propositions 1-2. In Point (a), as $|\phi| < \underline{\phi}$, the rule does not change the system's dynamics enough to affect the determinacy status, and this status remains the same as under a policy-instrument peg. Compared to Point (a) of Propositions 1-2, the novelty is that the determinacy status under a peg can now be not only M (when $q \leq \delta - 1$), but also D (when $q = \delta$) and E (when $q \geq \delta + 1$).

[7] The "no-decoupling condition" requires that the system should not be "decoupled" in the sense of Sims (2007). It is formulated as a matrix-rank condition in Blanchard and Kahn (1980, Page 1308), and is often called the "rank condition" in the literature. Sims' (2007) bare-bones example of a system meeting the root-counting condition but not the no-decoupling condition is $x_t = 1.1 x_{t-1} + \varepsilon_t$ and $\mathbb{E}_t\{y_{t+1}\} = 0.9 y_t + \nu_t$.

Figure 3: Determinacy status for systems with $q_{\mathcal{C}} = r_{\mathcal{C}} = 0$



In Point (b), as $|\phi| > \bar{\phi}$, the rule dominates the structural equations in the system's dynamics and makes the determinacy status depend only on h. Compared to Point (b) of Propositions 1-2, the novelty is that the horizon h^* can now be different from zero.

Points (c)-(i), (c)-(ii), (d)-(i), and (d)-(ii) of Proposition 5 generalize Point (d) of Propositions 1-2. For $|\phi| \in (\underline{\phi}, \bar{\phi})$, as $|h| \to +\infty$, the roots of $P(z)$ distribute themselves between inside and outside $\mathcal{C}$ in proportion of the share of $\mathcal{C}$ on which the structural equations dominate the rule and the share of $\mathcal{C}$ on which the rule dominates the structural equations. So, as $h \to -\infty$ (resp. as $h \to +\infty$), we eventually get more (resp. fewer) inside roots than non-predetermined variables, and hence explosiveness (resp. multiplicity).

Finally, Points (c)-(iii) and (d)-(iii) of Proposition 5 generalize Point (c) of Propositions 1-2. Large positive (resp. negative) horizons h do not much perturb the imploding (resp. exploding) equilibrium paths obtained under a policy-instrument peg, as the term $\mathbb{E}_t\{v_{t+h}\}$ is small on these paths; so, these horizons preserve the determinacy status obtained under a peg if this status is multiplicity (resp. explosiveness).

3.3 Numerical examples

In order to illustrate Proposition 5 and the next propositions numerically, I consider, in addition to Model 1, five other simple calibrated monetary-policy models. Table 1 presents the overall six models: two are with $q = \delta - 1$, two with $q = \delta$, and two with $q = \delta + 1$. All these models share the following features: they have two structural equations; these equations are an IS equation and a Phillips curve; and the two endogenous variables set by the private sector are output and inflation.

Table 1: Six simple calibrated monetary-policy models

No.	Model	Calibration	$q-\delta$
1	Basic New Keynesian Model	Woodford (2003)	-1
2	McKay et al. (2017)	McKay et al. (2017)	-1
3	Gabaix (2020)	Gabaix (2020)	0
4	Bilbiie (2008)	Bilbiie (2008)	0
5	Svensson (1997) and Ball (1999)	Ball (1999)	1
6	Rudebusch and Svensson (1999)	Rudebusch and Svensson (1999)	1

The IS equation and the Phillips curve of Models 2-4 are of type

$$
\begin{aligned}
y_t &= \alpha \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\{\pi_{t+1}\}\right), \\
\pi_t &= \beta \mathbb{E}_t\{\pi_{t+1}\} + \kappa y_t,
\end{aligned}
$$

where the notations are the same as in Section 2, and $\alpha \in (0,1]$ is an additional parameter. The parameters $(\alpha, \beta, \sigma, \kappa)$ take the following values: $(0.97, 0.99, 2.67, 0.02)$ in Model 2, $(0.85, 0.792, 5, 0.11)$ in Model 3, and $(1, 0.99, -0.11, 0.228)$ in Model 4. These models introduce, into the basic New Keynesian model, income risk and borrowing constraints (Model 2), bounded rationality (Model 3), or limited asset-markets participation (Model 4). Compared to the basic New Keynesian model (in which α is implicitly equal to 1), Model 2 "discounts" the IS equation (i.e. reduces α), Model 3 discounts both the IS equation and the Phillips curve (i.e. reduces both α and β), and Model 4 inverts the slope of the IS equation (i.e. makes σ negative).

Unlike Models 1-4, Models 5-6 are non-micro-founded and purely backward-looking. The IS equation and the Phillips curve of Model 5 are

$$
\begin{aligned}
y_t &= \lambda y_{t-1} - \mu\left(i_{t-1} - \pi_{t-1}\right), \\
\pi_t &= \pi_{t-1} + \chi y_{t-1},
\end{aligned}
$$

where $(\lambda, \mu, \chi) = (0.8, 1, 0.4)$. The IS equation and the Phillips curve of Model 6 have a richer lag structure:

$$
\begin{aligned}
y_t &= \sum\nolimits_{k=1}^{2} \lambda_k y_{t-k} - (\mu/4) \sum\nolimits_{k=1}^{4} \left(i_{t-k} - \pi_{t-k}\right), \\
\pi_t &= \sum\nolimits_{k=1}^{4} \theta_k \pi_{t-k} + \chi y_{t-1}.
\end{aligned}
$$

The parameters $(\lambda_1, \lambda_2, \mu, \theta_1, \theta_2, \theta_3, \theta_4, \chi)$, estimated on US data, take the values $(1.16, -0.25, 0.10, 0.70, -0.10, 0.28, 0.12, 0.14)$.

A first numerical illustration of Proposition 5 is provided by Figure 2 in Section 2. This figure, which I have already commented upon, shows the determinacy status $S(\phi, h)$ for Model 1 and Rules 1-2, in the pseudo half-plane $(h, \phi) \in \mathbb{Z} \times \mathbb{R}_+$ with a log scale for ϕ. Since Model 1 satisfies $q = \delta - 1$, Figure 2 is more specifically a numerical example of the left panel of Figure 3.

Figure 4 is the counterpart of Figure 2 for Models 3-5. The top four panels of this figure show the determinacy status for the systems (Model j, Rule k) with $j \in \{3, 4\}$ and $k \in \{1, 2\}$. These systems satisfy $q_{\mathcal{C}} = r_{\mathcal{C}} = 0$; therefore, they fall within the ambit of Proposition 5. Moreover, Models 3-4 satisfy $q = \delta$; so, the top four panels of Figure 4 are, more specifically, numerical examples of the middle panel of Figure 3. In these panels, thus, the rule can deliver determinacy for any horizon h, unlike in Figure 2. Compared to standard values of ϕ, the lower threshold $\underline{\phi}$ is of the same order of magnitude or several orders of magnitude smaller, while the upper threshold $\bar{\phi}$ is of the same order of magnitude or several orders of magnitude larger. The "topology" of the E, D, and M regions is simple in the top two panels (for Model 3): each region is connected, and the borders between regions are monotonic functions linking h to ϕ. The topology is more complex in the middle two panels (for Model 4): the M region is disconnected, and the borders between regions are non-monotonic, with "lace patterns."

The bottom left panel of Figure 4 shows the determinacy status for the system (Model 5, Rule 1). This system also satisfies $q_{\mathcal{C}} = r_{\mathcal{C}} = 0$, and hence also falls within the ambit of Proposition 5. Moreover, Model 5 satisfies $q = \delta + 1$; so, the bottom left panel of Figure 4 is, more specifically, a numerical example of the right panel of Figure 3. Qualitatively speaking, the D region in the bottom left panel of Figure 4 looks like the mirror image, with left and right reversed, of the D region in the left panel of Figure 2. The horizon threshold at and below which the rule can no longer deliver determinacy is minus one period; so, in this example, a central bank that would react to inflation with a delay of at least one period would necessarily be behind the curve and fail to ensure determinacy, no matter how strongly or weakly it reacts to inflation. Compared to standard values of ϕ, the lower threshold is of the same order of magnitude, while the upper threshold is one order of magnitude larger.

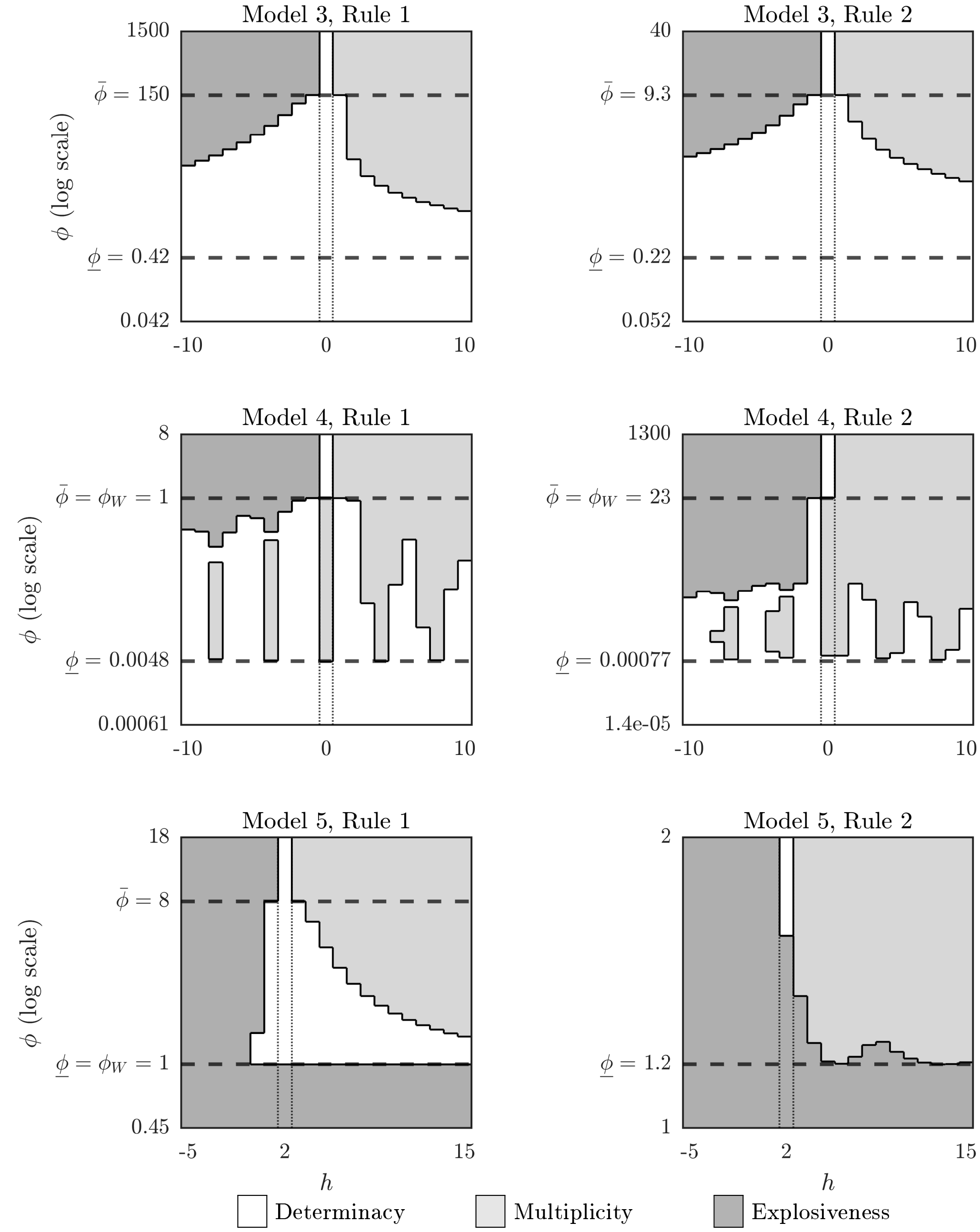
Finally, the bottom right panel of Figure 4 shows the determinacy status for the system (Model 5, Rule 2). This system satisfies $q_{\mathcal{C}} = 0$, but not $r_{\mathcal{C}} = 0$. For this system, indeed, we have $R(z) = (1 - z)/\chi$ and hence $r_{\mathcal{C}} = 1$, because the Phillips curve of Model 5 is vertical in the long run. So, the bottom right panel of Figure 4 is not an illustration of Proposition 5. I will analyze the case $r_{\mathcal{C}} \geq 1$ in Section 4.

3.4 Determinacy horizons

I now turn to the question of whether the set of determinacy horizons $H_D := \{h \in \mathbb{Z} | \exists \phi \in \mathbb{R}, S(\phi, h) = D\}$ is bounded or not, below or above. This question matters for the desirability of backward- or forward-looking stabilization policy. Let $A_{min} := \arg\min_{z \in \mathcal{C}} |Q(z)/R(z)|$. I provide the following answer to this question:

Proposition 6 (Determinacy horizons for systems with $q_{\mathcal{C}} = r_{\mathcal{C}} = 0$): *Consider a model (4) and a variable v_t such that $q_{\mathcal{C}} = r_{\mathcal{C}} = 0$. Then, under the rule $i_t = \phi \mathbb{E}_t \{v_{t+h}\}$ with $(\phi, h) \in \mathbb{R} \times \mathbb{Z}$:*
(a) if $q = \delta$, then $H_D = \mathbb{Z}$;
(b) if $q \leq \delta - 1$ (resp. $q \geq \delta + 1$), then: (i) H_D is bounded above (resp. below); (ii) if $q - \delta$

Figure 4: Determinacy status for Models 3-5 and Rules 1-2



is odd and $A_{min} \subset \mathcal{C} \setminus \{-1, 1\}$, then H_D is bounded below (resp. above); (iii) if $q = \delta - 1$ (resp. $q = \delta + 1$) and $A_{min} \in \{\{-1\}, \{1\}\}$, then H_D is unbounded below (resp. above).

Proof: See Appendix A.8. ■

Points (a) and (b)(i) of this proposition straightforwardly follow from Points (a), (c)(iii), and (d)(iii) of Proposition 5. Point (a) of Proposition 6 is illustrated in the top four panels of Figure 4, while Point (b)(i) of Proposition 6 is illustrated in the two panels of Figure 2 and the bottom left panel of Figure 4.

Point (b)(ii) of Proposition 6 states a sufficient condition for the absence of a D region either in the left tail of the left panel of Figure 3, or in the right tail of the right panel of Figure 3. This point generalizes, along several dimensions, Point (b) of Proposition 4. Under the condition stated in this point, as $|\phi|$ crosses $\underline{\phi}$, an even number of roots of $P(z)$ cross $\mathcal{C}$; in order to get determinacy, however, we would need an odd number of them. For $|h|$ sufficiently large, if no $|\phi|$ in the neighborhood of $\underline{\phi}$ can ensure determinacy, then more generally no $|\phi| \in \mathbb{R}_+$ can ensure determinacy. So, the determinacy status jumps directly from multiplicity to explosiveness, or vice-versa, as $|\phi|$ goes from zero to infinity. Under the condition stated in this point, thus, a sufficiently backward- or forward-looking stabilization policy will necessarily fail to deliver determinacy, no matter how strongly or weakly the policy instrument reacts to the state of the economy. This point is, again, illustrated in the right panel of Figure 2.

Finally, Point (b)(iii) of Proposition 6 states a sufficient condition for the existence of an unbounded D region either in the left tail of the left panel of Figure 3, or in the right tail of the right panel of Figure 3. This point generalizes Point (a) of Proposition 3 and Point (b) of Proposition 4. Under the condition stated in this point, as $|\phi|$ crosses $\underline{\phi}$, we need exactly one root of $P(z)$ to cross $\mathcal{C}$ in order to get determinacy, and we do get exactly one such root (for all horizons if $A_{min} = \{1\}$, and for every other horizon if $A_{min} = \{-1\}$). We also need this root to cross $\mathcal{C}$ in the right direction, and the root does so for h or $-h$ sufficiently large (depending on the sign of $q - \delta$). Thus, when a policy-instrument peg generates multiplicity (resp. explosiveness), the rule can be arbitrarily backward-looking (resp. forward-looking) and still ensure determinacy; as we know from Points (c)(ii) and (d)(ii) of Proposition 5, however, the set of values for ϕ leading to determinacy gradually shrinks to the empty set as $h \to -\infty$ (resp. $h \to +\infty$). These results are illustrated in the left panel of Figure 2 and the bottom left panel of Figure 4.

3.5 Taylor principle

Finally, I investigate the validity of the Taylor principle as a condition for determinacy. I start by providing a formal, general definition of Woodford's (2001, 2003) long-run Taylor principle. Woodford (2001, 2003) mostly describes this principle in the specific context of the basic New Keynesian model with several alternative parametric families of interest-rate rules. He discusses how to generalize this principle to a broader context as follows: "*One observes quite generally – in the case of* any *family of policy rules that involve feedback only from inflation and output, regardless of how many lags of these might be involved – that the boundary between sets of coefficients that satisfy the Taylor principle and those that do not will consist of coefficients for which there is an eigenvalue exactly equal to* 1. *(...) It follows that a real eigenvalue crosses the unit circle as the sign of the inequality corresponding to the Taylor principle changes. This boundary is therefore one at which the number of unstable eigenvalues increases by one. Often this results in moving from a situation of indeterminacy to determinacy, though I do not seek to* *establish general conditions for this*" (Woodford, 2003, Chapter 4, Page 256, Footnote 27).

The system (4)-(5) has an eigenvalue equal to 1 if and only if $P(1) = 0$. If $R(1) = 0$, then $P(1)$ does not depend on the coefficient ϕ. Alternatively, if $R(1) \neq 0$, then $P(1) = 0$ if and only if $\phi = \phi_W := -Q(1)/R(1)$ (where again the subscript W stands for "Woodford"). In all the examples considered by Woodford (2001, 2003), ϕ_W is non-negative and the Taylor principle is $\phi > \phi_W$, not $\phi < \phi_W$ (i.e., the policy instrument should react to the variable sufficiently strongly, not sufficiently weakly). So, I propose the following definition of the Taylor principle:

Definition 1 (Taylor principle): *If $R(1) \neq 0$ and $\phi_W := -Q(1)/R(1) \geq 0$, then the Taylor principle is $\phi > \phi_W$.*

This definition is a generalization of the definition considered in Section 2. The latter definition was tailored to the specific context of the basic New Keynesian model. Both definitions lead to the same Taylor principle in this model: $\phi > 1$ for Rule 1, and $\phi > \kappa/(1-\beta)$ for Rule 2. The definition in Section 2 referred to the permanent reaction of the interest rate to a permanent change in inflation; the eigenvalue 1 in Woodford's quote and the root 1 of $P(z)$ capture these long-run changes. The advantage of Definition 1 is that it applies not just to the basic New Keynesian model with inflation or output in the interest-rate rule, but more generally to any stabilization-policy model and any variable v_t in the policy-instrument rule $-$ as long as ϕ_W exists and is non-negative. In fact, in the literature, one can already come across Definition 1's Taylor principle outside the context of the basic New Keynesian model: the "income-risk augmented Taylor principle" of Acharya and Dogra (2020) and the "HANK Taylor principle" of Bilbiie (2021), for instance, coincide with Definition 1's Taylor principle (as long as $\phi_W \geq 0$).

The condition that ϕ_W should exist and be non-negative is, of course, not always met. The system (Model 5, Rule 2), for instance, is such that ϕ_W does not exist, since $R(1) = 0$ in this system (as discussed in Subsection 3.3). The systems (Model 3, Rule 1) and (Model 3, Rule 2) are such that ϕ_W exists but is negative; in this case, whether the Taylor principle should be defined as $\phi > \phi_W$ or $\phi < \phi_W$ is unclear. These three systems correspond to the top left, top right, and bottom right panels of Figure 4; so, ϕ_W does not appear in these panels. I have featured ϕ_W in the other panels of this figure.

Let $A_{max} := \arg\max_{z \in \mathcal{C}} |Q(z)/R(z)|$. I can now state the results about the Taylor principle as follows:

Proposition 7 (Taylor principle for systems with $q_{\mathcal{C}} = r_{\mathcal{C}} = 0$): *Consider a model (4) and a variable v_t such that $q_{\mathcal{C}} = r_{\mathcal{C}} = 0$ and $\phi_W > 0$. Let $h^{**} := m + R'(1)/R(1) - Q'(1)/Q(1)$. Then, under the rule $i_t = \phi \mathbb{E}_t \{v_{t+h}\}$ with $(\phi, h) \in \mathbb{R}_+ \times \mathbb{Z}$:*

(a) if $q \neq \delta$, then $\forall h \in \mathbb{Z} \setminus H_D$, the Taylor principle is irrelevant for D;

(b) if $q = \delta$, then $\forall h \in \mathbb{Z}$, the Taylor principle is not necessary for D;

(c) $\forall h \in \mathbb{Z} \setminus \{h^*\}$*, the Taylor principle is not sufficient for D;*

(d) if $A_{max} = \{1\}$*, then: (i)* $\phi_W = \bar{\phi}$*, (ii)* $\forall h \leq h^* - 1$*, the Taylor principle is sufficient for E, (iii) for* $h = h^*$*, it is sufficient for D, (iv)* $\forall h \geq h^* + 1$*, it is sufficient for M;*

(e) if $A_{min} = \{1\}$*, then: (i)* $\phi_W = \underline{\phi}$*, (ii)* $\forall h \in \mathbb{Z}$*, the Taylor principle is locally necessary and sufficient for D if and only if (*$q = \delta - 1$ *and* $h < h^{**}$*) or (*$q = \delta + 1$ *and* $h > h^{**}$*).*

Proof: See Appendix A.9. ■

Points (a)-(d) of this proposition straightforwardly follow from the definition of H_D and Points (a)-(b) of Proposition 5. Point (d) of Proposition 7, in particular, is illustrated in the right panel of Figure 2 and the middle two panels of Figure 4.

Point (e) of Proposition 7 states the necessary and sufficient condition for the Taylor principle to be locally necessary and sufficient for determinacy when $\phi_W = \underline{\phi}$. This point generalizes Point (c) of Proposition 3 and Point (f) of Proposition 4. If $q = \delta - 1$ (resp. $q = \delta + 1$), then, as ϕ crosses $\phi_W = \underline{\phi}$ from below, we need exactly one root of $P(z)$ to go from outside to inside $\mathcal{C}$ (resp. from inside to outside $\mathcal{C}$) in order to get determinacy, and we do get exactly one such root if and only if $h < h^{**}$ (resp. $h > h^{**}$). Alternatively, if $|q - \delta| \neq 1$, then we still have exactly one root of $P(z)$ crossing $\mathcal{C}$ as ϕ crosses $\phi_W = \underline{\phi}$, but we would need a different number of such roots in order to get determinacy. This point is illustrated in the left panel of Figure 2 and the bottom left panel of Figure 4.

Points (d)-(e) of Proposition 7 focus on the cases in which $\phi_W \in \{\underline{\phi}, \bar{\phi}\}$. However, the case in which $\phi_W \in (\underline{\phi}, \bar{\phi})$ may also arise; the system (Model 2, Rule 2) is one example (figure not shown).

4 Determinacy analysis for non-regular systems

In this section, I consider the same class of models and the same class of rules as in the previous section, and I extend the analysis to non-regular systems, i.e. systems with either $q_{\mathcal{C}} \geq 1$ or $r_{\mathcal{C}} \geq 1$. I highlight which results do not change and which ones do, and how and why, as we move from regular to non-regular systems.

4.1 Sources of non-regularity

I start with a brief discussion of the sources of non-regularity. To that aim, and also to illustrate the results that I will obtain for non-regular systems, I consider, in addition to Rules 1 and 2, four other simple interest-rate rules. The overall six rules are presented in Table 2. Consistently with the analysis so far, they make the interest rate react to only one variable (or linear combination of variables), at horizon h and with coefficient ϕ. I will consider policy-instrument rules with several variables, or with policy-instrument inertia, in Section 5.

Table 2: Six simple interest-rate rules

No.	Rule	Variable in the rule
1	$i_t = \phi \mathbb{E}_t\{\pi_{t+h}\}$	Inflation
2	$i_t = \phi \mathbb{E}_t\{y_{t+h}\}$	Real Output
3	$i_t = \phi \mathbb{E}_t\{\Delta y_{t+h}\}$	Real-Output Growth
4	$i_t = \phi \mathbb{E}_t\{p_{t+h}\}$	Price Level
5	$i_t = \phi \mathbb{E}_t\{p_{t+h} + y_{t+h}\}$	Nominal Output
6	$i_t = \phi \mathbb{E}_t\{\pi_{t+h} + \Delta y_{t+h}\}$	Nominal-Output Growth

The six variables in the six rules are standard in the monetary-policy literature. Rules 1 and 2 are the rules I considered in the previous sections; they involve inflation and (real) output respectively, and can be viewed as special cases of what are often called (backward-, current-, or forward-looking) "Taylor rules," after Taylor (1993). Rule 4 involves the price level (denoted by p_t at date t), and corresponds to what Woodford (2003) calls the "Wicksellian rule," after Wicksell (1898). Rule 5 involves nominal output; like inflation and the price level, nominal output is a variable that has a long history as a candidate target for monetary policy. Note that the variables in Rules 1, 3, and 6 are the first differences of the variables in Rules 4, 2, and 5 respectively. The question of whether the monetary-policy instrument should react to variables in levels or in growth rates is a long-standing issue in the literature (see, e.g., McCallum, 1999).

Six models times six rules makes thirty-six systems. Table 3 displays the values of $q_{\mathcal{C}}$ and $r_{\mathcal{C}}$ for these systems. More than half of them are non-regular.

Table 3: $q_{\mathcal{C}}$ and $r_{\mathcal{C}}$ for Models 1-6 and Rules 1-6

Model	Rule 1		Rule 2		Rule 3		Rule 4		Rule 5		Rule 6	
	$q_{\mathcal{C}}$	$r_{\mathcal{C}}$	$q_{\mathcal{C}}$	$r_{\mathcal{C}}$	$q_{\mathcal{C}}$	$r_{\mathcal{C}}$	$q_{\mathcal{C}}$	$r_{\mathcal{C}}$	$q_{\mathcal{C}}$	$r_{\mathcal{C}}$	$q_{\mathcal{C}}$	$r_{\mathcal{C}}$
1	0	0	0	0	0	1	1	0	1	0	0	0
2	0	0	0	0	0	1	1	0	1	0	0	0
3	0	0	0	0	0	1	1	0	1	0	0	0
4	0	0	0	0	0	1	1	0	1	0	0	0
5	0	0	0	1	0	2	1	0	1	2	0	2
6	0	3	0	4	0	5	1	3	1	3	0	3

Note: When $q_{\mathcal{C}} = 1$, the root of $Q(z)$ on $\mathcal{C}$ is 1. When $r_{\mathcal{C}} \geq 1$, the set of roots of $R(z)$ on $\mathcal{C}$, counting multiplicity, is $\{1\}$, $\{1,1\}$, $\{0.8+0.6i, 0.8-0.6i\}$, $\{-1, i, -i\}$, $\{1, -1, i, -i\}$, or $\{1, 1, -1, i, -i\}$.

Systems with $r_{\mathcal{C}} \geq 1$ can obtain under several alternative circumstances. First, $R(z)$ is a multiple of $1-z$ (implying $R(1) = 0$ and $r_{\mathcal{C}} \geq 1$) when the rule makes the policy instrument react to a variable in first difference, rather than in level. For instance, all the systems (Model j, Rule 3) for $j \in \{1, ...6\}$ are of that kind, as Rule 3 makes the interest rate react to the output growth rate, rather than the output level like Rule 2. In particular, in the basic New Keynesian model considered in Section 2 (whose calibrated version is Model 1), we had $R(z) = (z-\beta)/\sigma$ under Rule 2; under Rule 3, we have $R(z) = (1-z)(z-\beta)/\sigma$, and hence $R(1) = 0$ and $r_{\mathcal{C}} = 1$.

Second, in monetary-policy contexts, $R(z)$ is also a multiple of $1-z$ under Rule 2 when the long-run Phillips curve is vertical, as in Models 5-6. In particular, in the model of Svensson (1997) and Ball (1999) described in Section 3 (whose calibrated version is Model 5), we have $R(z) = (1-z)\mu/\chi$ under Rule 2, and hence $R(1) = 0$ and $r_{\mathcal{C}} = 1$.

Third, $R(z)$ is a multiple of $\sum_{k=0}^{T-1} z^k$ with $T \geq 2$ (implying $r_{\mathcal{C}} \geq 1$) when the structural equations involve the average value of the policy instrument over T dates, rather than the policy instrument at a single date. For instance, in the model of Rudebusch and Svensson (1999) described in Section 3 (whose estimated version is Model 6), the IS equation involves the term $(1/4)\sum_{k=1}^{4} i_{t-k}$, which can be interpreted as an ex-post medium-term interest rate; as a result, for any interest-rate rule, $R(z)$ is a multiple of $\sum_{k=0}^{3} z^k$, so $R(-1) = R(-i) = R(i) = 0$ and $r_{\mathcal{C}} \geq 3$. Fourth, and relatedly, $R(z)$ is also a multiple of $\sum_{k=0}^{T-1} z^k$ with $T \geq 2$ when the variable v_t in the rule is an average over T dates – like, e.g., four-quarter average inflation in Levin et al. (1999, 2003) and Levin and Williams (2003), or twelve-quarter average inflation in Taylor and Williams (2011).

Systems with $q_{\mathcal{C}} \geq 1$, in monetary-policy contexts, typically obtain when the structural equations involve the inflation rate but not the price level per se (as in Models 1-6), while the interest-rate rule involves the price level but not the inflation rate (like Rules 4-5). In this case, the dynamic equation has typically to be expressed in terms of the price level, not the inflation rate, and $Q(z)$ is then a multiple of $1-z$. In the basic New Keynesian model considered in Section 2, for instance, we had $Q(z) = \beta - (1+\beta+\kappa/\sigma)z + z^2$ under Rule 1; under Rule 4, we have $Q(z) = [\beta - (1+\beta+\kappa/\sigma)z + z^2](z-1)$, and hence $Q(1) = 0$ and $q_{\mathcal{C}} = 1$.

In the following, I distinguish between three kinds of non-regular system: the systems with $q_{\mathcal{C}} = 0$ and $r_{\mathcal{C}} \geq 1$, those with $q_{\mathcal{C}} \geq 1$ and $r_{\mathcal{C}} = 0$, and those with $q_{\mathcal{C}} \geq 1$ and $r_{\mathcal{C}} \geq 1$. I extend the analysis of Section 3 to each kind of non-regular system, and I illustrate the results with determinacy-status figures for some of the thirty-six systems (selected for their illustrative value). I focus again on positive values of ϕ in these figures, for consistency with the theoretical and empirical literatures.

4.2 Systems with $q_{\mathcal{C}} = 0$ and $r_{\mathcal{C}} \geq 1$

I start with the case in which $q_{\mathcal{C}} = 0$ and $r_{\mathcal{C}} \geq 1$. I have mentioned above four alternative circumstances under which this case may arise; under two of them, we typically get, more specifically, $r_{\mathcal{C}} = 1$ and $R(1) = 0$; so, I will pay particular attention to this subcase. I obtain the following results:

Proposition 8 (Determinacy status, determinacy horizons, and Taylor principle for systems with $q_{\mathcal{C}} = 0$ and $r_{\mathcal{C}} \geq 1$): *Consider a model (4) and a variable v_t such that $q_{\mathcal{C}} = 0$ and $r_{\mathcal{C}} \geq 1$. Let $\underline{\phi} := \min_{z\in\mathcal{C}} |Q(z)/R(z)|$, $\underline{h}^* := m + r - \delta$, and $\bar{h}^* := \underline{h}^* + r_{\mathcal{C}}$. Then there exists*

$\bar{\phi} > 0$ such that, under the rule $i_t = \phi \mathbb{E}_t \{v_{t+h}\}$ with $(\phi, h) \in \mathbb{R} \times \mathbb{Z}$, Points (a) and (c)-(d) of Proposition 5 still hold and:

(b1) $\forall |\phi| > \bar{\phi}$, (i) $\forall h \leq \underline{h}^* - 1$, $S(\phi, h) = E$, (ii) $\forall h \in \{\underline{h}^*, ..., \bar{h}^*\}$, $S(\phi, h)$ may depend on ϕ only through the sign of ϕ, (iii) $\forall h \geq \bar{h}^* + 1$, $S(\phi, h) = M$;

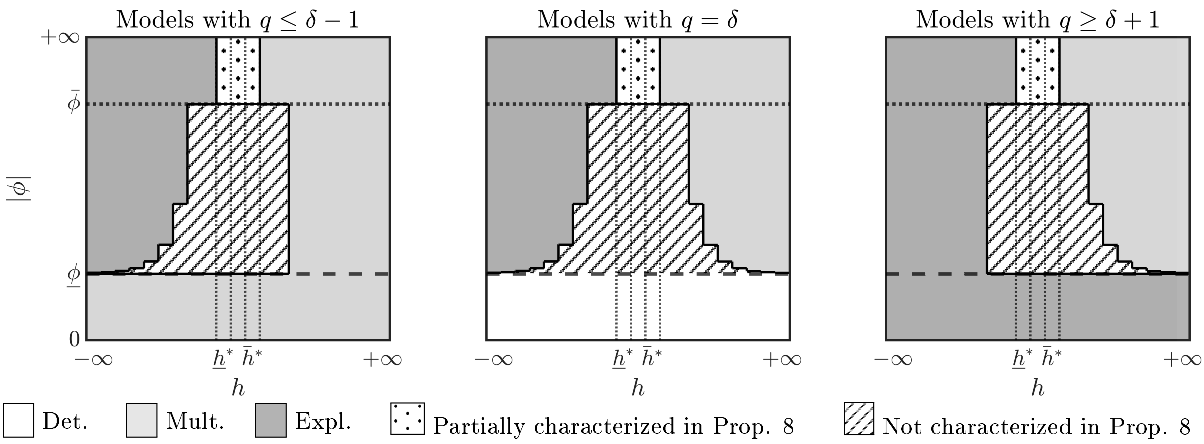
(b2) if $r_{\mathcal{C}} = 1$ and $R(1) = 0$, then: (i) if $Q(1)R'(1) > 0$ then $\forall \phi > \bar{\phi}$, $(S(\phi, \underline{h}^*), S(\phi, \bar{h}^*)) = (E, D)$ and $(S(-\phi, \underline{h}^*), S(-\phi, \bar{h}^*)) = (D, M)$, (ii) if $Q(1)R'(1) < 0$ then $\forall \phi > \bar{\phi}$, $(S(\phi, \underline{h}^*), S(\phi, \bar{h}^*)) = (D, M)$ and $(S(-\phi, \underline{h}^*), S(-\phi, \bar{h}^*)) = (E, D)$.

In addition, Points (a)-(b) of Proposition 6 still hold and, if $R(1) \neq 0$ and $\phi_W > 0$, Points (a)-(b) and (e) of Proposition 7 still hold as well.

Proof: See Appendix A.10. ■

The determinacy-status results stated in Proposition 8 are diagrammatically summarized in Figure 5. There are two changes relatively to Proposition 5 and Figure 3. First, $\bar{\phi}$ is no longer defined as $\max_{z \in \mathcal{C}} |Q(z)/R(z)|$, since this maximum no longer exists (as $R(z)$ has at least one root on $\mathcal{C}$). Proposition 8 does not provide any expression for the new $\bar{\phi}$. To highlight this change, I represent the horizontal line $\phi = \bar{\phi}$ as a dotted line in Figure 5, rather than a dashed line (as in Figure 3). The second change is that for $|\phi| > \bar{\phi}$, the single horizon h^* that led to determinacy, in Point (b) of Proposition 5, has been replaced, in Points (b1)-(b2) of Proposition 8, by a range of $r_{\mathcal{C}} + 1$ horizons $\{\underline{h}^*, ..., \bar{h}^*\}$ that may or may not lead to determinacy. As $|\phi| \to +\infty$, some roots of $P(z)$ converge to the $r_{\mathcal{C}}$ roots of $R(z)$ on $\mathcal{C}$; some may converge from inside $\mathcal{C}$, others from outside; hence the range of horizons $\{\underline{h}^*, ..., \bar{h}^*\}$ for which we may or may not get determinacy.

Figure 5: Determinacy status for systems with $q_{\mathcal{C}} = 0$ and $r_{\mathcal{C}} \geq 1$



As far as determinacy horizons are concerned, there is no change relatively to Proposition 6. The reason is that whether the set H_D is bounded or not, either below or above, only depends on the properties of the system for $|\phi|$ in the neighborhood of $\underline{\phi}$, and these properties do not depend on whether $r_{\mathcal{C}} = 0$ or $r_{\mathcal{C}} \geq 1$. Similarly, regarding the Taylor principle, Points (a)-(b) and (e) of Proposition 7 still hold because they only rest on the properties of the system for $|\phi|$ lower than $\underline{\phi}$ or in the neighborhood of $\underline{\phi}$, and these properties do not depend on whether $r_{\mathcal{C}} = 0$ or $r_{\mathcal{C}} \geq 1$. Points (c)-(d) of Proposition 7, on the contrary, no longer hold. The reason is that these points rest on the properties of the system for $|\phi|$ higher than $\bar{\phi}$ or in the neighborhood of $\bar{\phi}$, and these properties have changed. Instead of Points (c)-(d) of Proposition 7, we now have that the Taylor principle is not sufficient for any $h \in \mathbb{Z} \setminus \{\underline{h}^*, ..., \bar{h}^*\}$ and that, if $r_{\mathcal{C}} = 1$ and $R(1) = 0$, it is not sufficient either for $h = \underline{h}^*$ (if $Q(1)R'(1) > 0$) or for $h = \bar{h}^*$ (if $Q(1)R'(1) < 0$). These results are not stated explicitly in Proposition 8, but follow straightforwardly from Points (b1)-(b2) of this proposition.

Proposition 8 is illustrated in the bottom right panel of Figure 4 and the six panels of Figure 6. The latter figure shows the determinacy status for the systems (Model j, Rule 3) with $j \in \{1, ..., 6\}$. In all these seven panels, I do not draw any horizontal line $\phi = \bar{\phi}$ because I have no expression for $\bar{\phi}$; and I cannot feature ϕ_W because it does not exist (since $R(1) = 0$).

In the bottom right panel of Figure 4 and the top four panels of Figure 6, we have $r_{\mathcal{C}} = 1$ and $R(1) = 0$. So, Point (b2) of Proposition 8 applies, and there is a single horizon at which determinacy obtains for a sufficiently high coefficient ϕ. In the bottom left panel of Figure 6, we have $r_{\mathcal{C}} = 2$ and we get a single such horizon too. In the bottom right panel, we have $r_{\mathcal{C}} = 5$ and we get no such horizon.

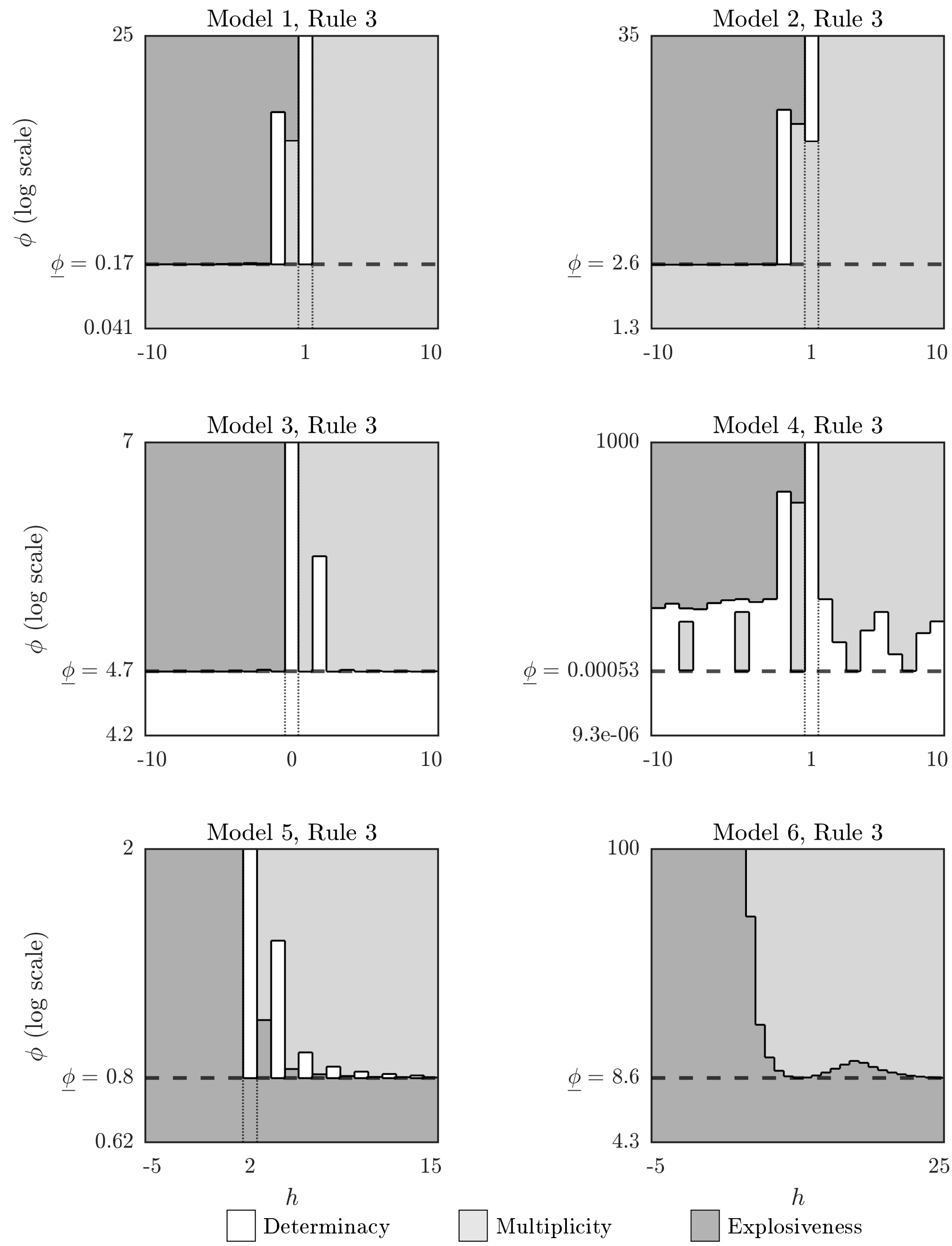
Now consider the set of horizons delivering determinacy for at least one non-negative value of the coefficient ϕ: $H_D^+ := \{h \in \mathbb{Z} | \exists \phi \in \mathbb{R}_+, S(\phi, h) = D\}$, which is a subset of H_D. In the bottom right panels of Figures 4 and 6, this set is bounded below and above as a consequence of Points (b)(i) and (b)(ii) of Proposition 6; the latter point applies because $A_{min} \subset \mathcal{C} \setminus \{-1, 1\}$ in these panels. In the top two panels (resp. the bottom left panel) of Figure 6, H_D^+ is bounded above (resp. below) as a result of Point (b)(i) of Proposition 6. We have $A_{min} = \{-1\}$ in these panels, so Point (b)(iii) of Proposition 6 applies, and H_D^+ is unbounded below (resp. above) – although this result is not visible in some panels, due to their scale. In the middle two panels of Figure 6, the set H_D^+ is simply $\mathbb{Z}$, as a result of Point (a) of Proposition 5.

Interestingly, there does not seem to be any determinacy region at all in the bottom right panel of Figure 6. For this system, I get no apparent determinacy region also when the coefficient ϕ is negative (figure not shown). So, Rule 3 apparently fails to ensure determinacy for any $\phi \in \mathbb{R}$ and any $h \in \mathbb{Z}$ in the realm of Model 6. The same result obtains for Rule 2 (figure not shown).

4.3 Systems with $q_{\mathcal{C}} \geq 1$ and $r_{\mathcal{C}} = 0$

I now turn to the case in which $q_{\mathcal{C}} \geq 1$ and $r_{\mathcal{C}} = 0$. In Subsection 4.1, I have mentioned only one circumstance under which this case may arise; under this circumstance, we get more specifically $q_{\mathcal{C}} = 1$ and $Q(1) = 0$; so, I will pay particular attention to this subcase. Let me first define two new variables, S^+ and S^-, as follows: if there exists a function $\varepsilon : \mathbb{Z} \rightarrow \mathbb{R}_+ \setminus \{0\}$ such that

Figure 6: Determinacy status for Models 1-6 and Rule 3



$S(\phi, h)$ is constant on $\{(\phi, h)|h \in \mathbb{Z}, \phi \in (0, \varepsilon(h))\}$ (resp. $\{(\phi, h)|h \in \mathbb{Z}, \phi \in (-\varepsilon(h), 0)\}$), then S^+ (resp. S^-) takes this constant value; otherwise, S^+ (resp. S^-) takes the value NA (for "not available"). I can now state the results in the following way:

Proposition 9 (Determinacy status, determinacy horizons, and Taylor principle for systems with $q_{\mathcal{C}} \geq 1$ and $r_{\mathcal{C}} = 0$): *Consider a model (4) and a variable v_t such that $q_{\mathcal{C}} \geq 1$ and $r_{\mathcal{C}} = 0$. Let $\underline{\phi} := \min_{z \in \mathcal{C}} |Q(z)/R(z)| = 0$ and $\bar{\phi} := \max_{z \in \mathcal{C}} |Q(z)/R(z)|$. Then, under the rule $i_t = \phi \mathbb{E}_t \{v_{t+h}\}$ with $(\phi, h) \in \mathbb{R} \times \mathbb{Z}$, Points (b), (c)(i), (c)(ii), (d)(i), and (d)(ii) of Proposition*

5 and Points (c)-(d) of Proposition 7 still hold. In addition, if $q_{\mathcal{C}} = 1$ and $Q(1) = 0$, then:

(e1) if $q \leq \delta - 2$, then $(S^+, S^-) = (M, M)$;

(e2) if $q = \delta - 1$ and $Q'(1)R(1) > 0$ (resp. < 0), then $(S^+, S^-) = (D, M)$ (resp. (M, D));

(e3) if $q = \delta$ and $Q'(1)R(1) > 0$ (resp. < 0), then $(S^+, S^-) = (E, D)$ (resp. (D, E));

(e4) if $q \geq \delta + 1$, then $(S^+, S^-) = (E, E)$;

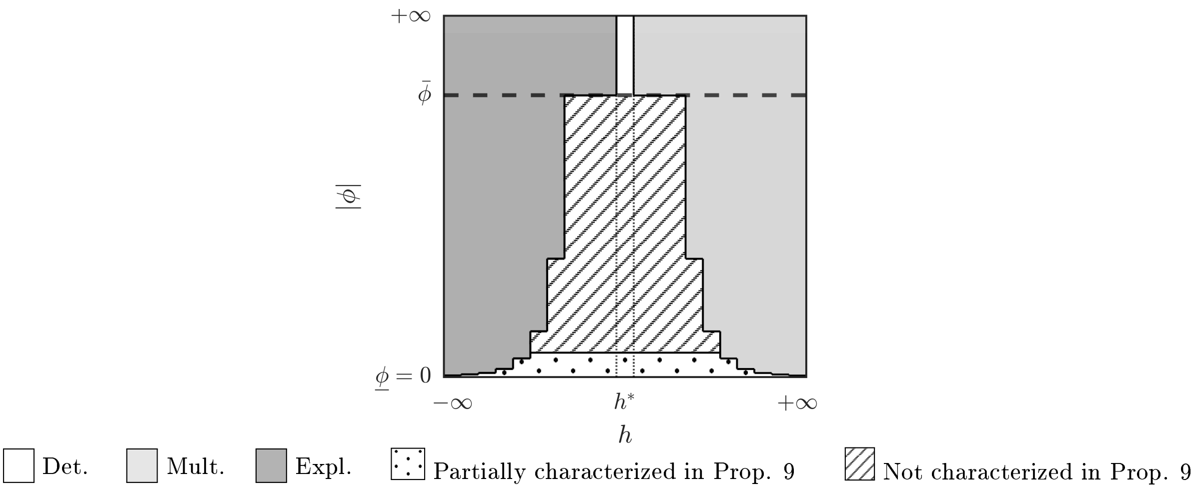
(e5) if $q \in \{\delta - 1, \delta\}$, then $H_D = \mathbb{Z}$;

(e6) (i) $\phi_W = 0$, (ii) $\forall h \in \mathbb{Z}$, the Taylor principle is locally necessary and sufficient for D if and only if ($q = \delta - 1$ and $Q'(1)R(1) > 0$) or ($q = \delta$ and $Q'(1)R(1) < 0$).

Proof: See Appendix A.13. ■

Proposition 9 is partly summarized in diagrammatic form in Figure 7. The changes, relatively to Proposition 5 and Figure 3, are due to the fact that now $\underline{\phi} := \min_{z \in \mathcal{C}} |Q(z)/R(z)| = 0$ (since $Q(z)$ has at least one root on $\mathcal{C}$). So, the case in which $|\phi| < \underline{\phi}$ can no longer arise; moreover, the properties of the system for $|\phi|$ just above $\underline{\phi}$ are modified.

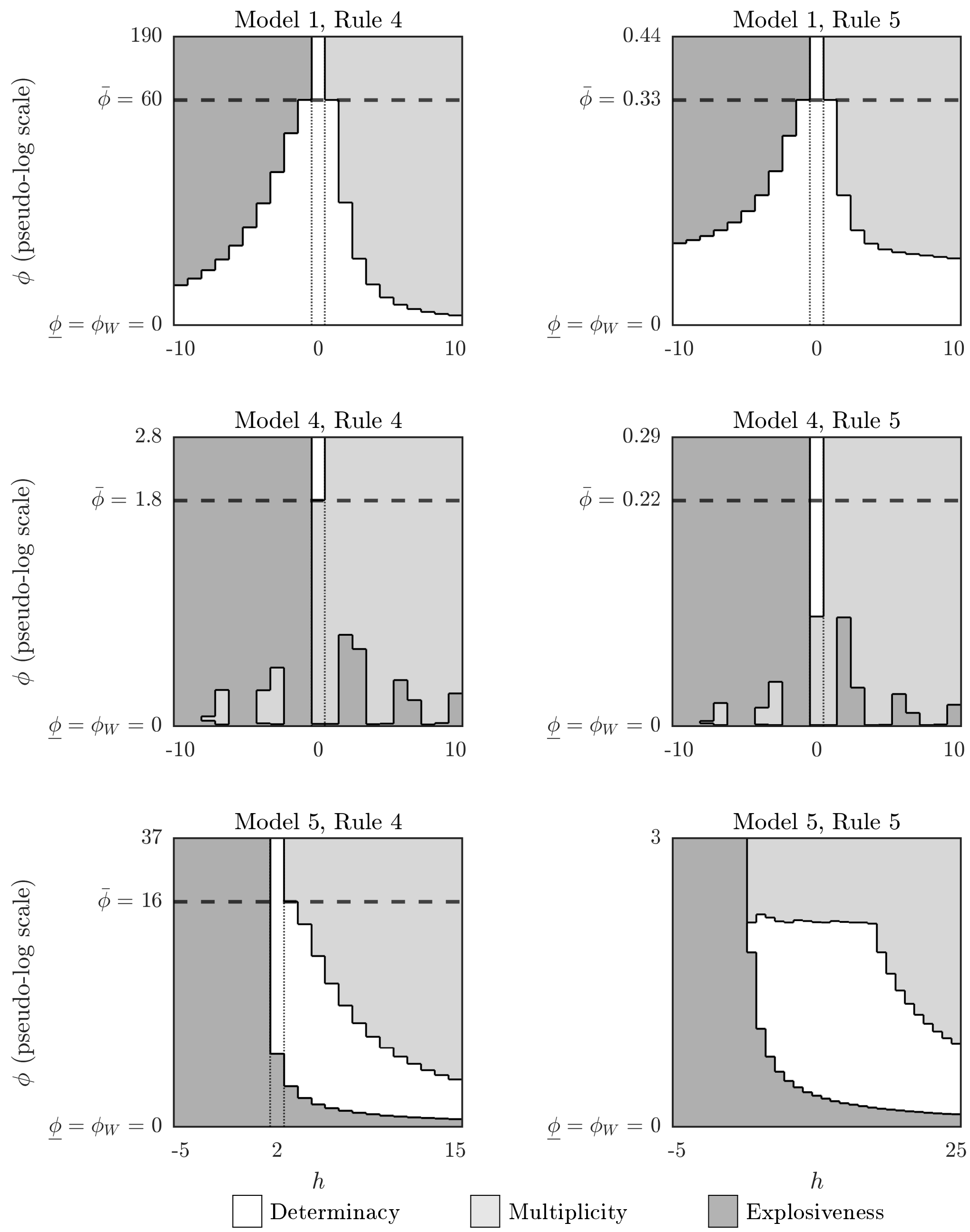
Figure 7: Determinacy status for systems with $q_{\mathcal{C}} \geq 1$ and $r_{\mathcal{C}} = 0$



Points (e5)-(e6) of Proposition 9, about the determinacy horizons and the Taylor principle, are a straightforward consequence of Points (e1)-(e4) of this proposition. In turn, the latter points can be understood as follows. If $q_{\mathcal{C}} = 1$ and $Q(1) = 0$, then moving from $\phi = 0$ to $\phi \neq 0$ moves a root of $P(z)$ from the point 1 on $\mathcal{C}$ to inside or outside $\mathcal{C}$. So, we get determinacy if we were missing one root inside $\mathcal{C}$ (i.e. $q = \delta - 1$) and the root leaving $\mathcal{C}$ moves inside $\mathcal{C}$ (i.e. $Q'(1)R(1)\phi > 0$), or if we had the right number of roots inside $\mathcal{C}$ (i.e. $q = \delta$) and the root leaving $\mathcal{C}$ moves outside $\mathcal{C}$ (i.e. $Q'(1)R(1)\phi < 0$). Alternatively, the determinacy status remains multiplicity if $q \leq \delta - 2$, and it remains explosiveness if $q \geq \delta + 1$.

I illustrate Proposition 9 with Figure 8. This figure shows the determinacy status for the systems (Model j, Rule k) with $j \in \{1, 4, 5\}$ and $k \in \{4, 5\}$. Like previously, I focus on the pseudo half-plane $(h, \phi) \in \mathbb{Z} \times \mathbb{R}_+$. Unlike previously, however, I use a pseudo-log scale for ϕ on the vertical axis rather than a log scale, i.e. I use the scale $\log(1+\phi)$ instead of $\log(\phi)$, in order to display the determinacy status in the neighborhood of $\phi = 0$.

Figure 8: Determinacy status for Models 1 and 4-5 and Rules 4-5



As reported in Table 3, we have $q_{\mathcal{C}} \geq 1$ and $r_{\mathcal{C}} = 0$ in all but one panel of Figure 8. The exception is the bottom right panel, in which $q_{\mathcal{C}} \geq 1$ and $r_{\mathcal{C}} \geq 1$; I will study this case in the next subsection. In the other panels of Figure 8, we have more specifically $q_{\mathcal{C}} = 1$ and $Q(1) = 0$, so Points (e1)-(e6) of Proposition 9 apply.

The top two panels of Figure 8 illustrate Points (e2) and (e5)-(e6) of Proposition 9: in these two panels, we have $q = \delta - 1$ and $Q'(1)R(1) > 0$, so we get determinacy for ϕ just above 0 ($S^+ = D$). Moreover, we get multiplicity for ϕ just below 0 ($S^- = M$, figure not shown). So, the Taylor principle is locally necessary and sufficient for determinacy in Model 1 under Rules 4-5.

The middle two panels of Figure 8 illustrate Point (e3) of Proposition 9: in these two panels, we have $q = \delta$ and $Q'(1)R(1) > 0$, so we get explosiveness for ϕ just above 0 ($S^+ = E$). Moreover, we get determinacy for ϕ just below 0 ($S^- = D$, figure not shown). So, the Taylor principle is locally necessary and sufficient for explosiveness in Model 4 under Rules 4-5; and it is locally necessary and sufficient for determinacy in the same model under the "opposite rules" $i_t = \phi \mathbb{E}_t\{-p_{t+h}\}$ and $i_t = \phi \mathbb{E}_t\{-p_{t+h} - y_{t+h}\}$.

Finally, the bottom left panel of Figure 8 illustrates Point (e4) of Proposition 9: in this panel, we have $q \geq \delta + 1$, so we get explosiveness for ϕ just above 0 ($S^+ = E$). We also get explosiveness for ϕ just below 0 ($S^- = E$, figure not shown).

4.4 Systems with $q_{\mathcal{C}} \geq 1$ and $r_{\mathcal{C}} \geq 1$

Finally, I consider systems with both $q_{\mathcal{C}} \geq 1$ and $r_{\mathcal{C}} \geq 1$. I restrict the analysis to the case in which $Q(z)$ and $R(z)$ have no common root on $\mathcal{C}$. In the alternative case, $P(z)$ would have at least one root on $\mathcal{C}$ for any $(\phi, h) \in \mathbb{R} \times \mathbb{Z}$, so the determinacy-status analysis would be inconclusive. In the light of the discussion in Subsection 4.1, it is anyway unclear how such a case could arise in the first place.

I obtain the following results:

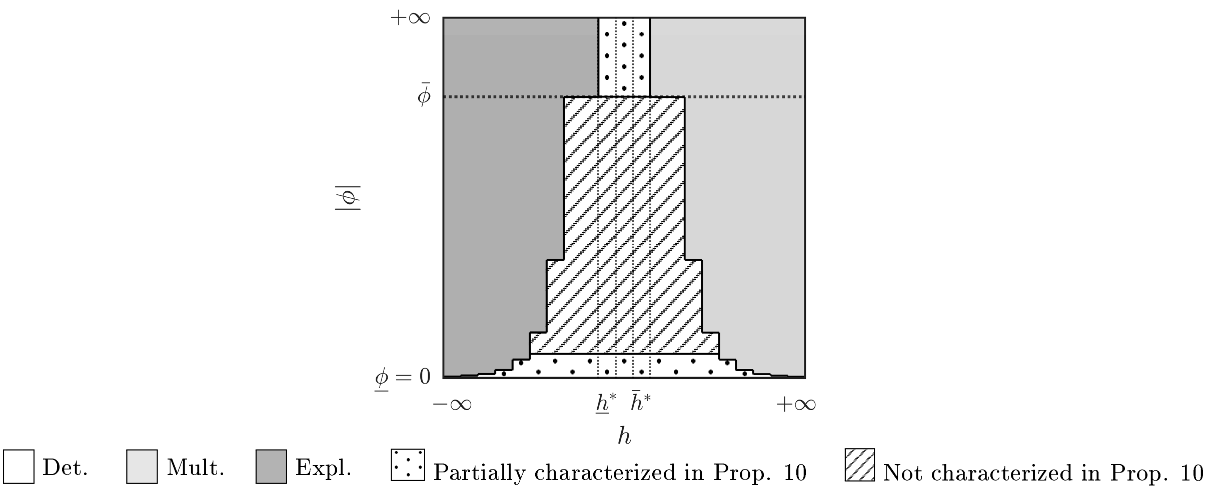
Proposition 10 (Determinacy status, determinacy horizons, and Taylor principle for systems with $q_{\mathcal{C}} \geq 1$ and $r_{\mathcal{C}} \geq 1$): *Consider a model (4) and a variable v_t such that $q_{\mathcal{C}} \geq 1$ and $r_{\mathcal{C}} \geq 1$, with $Q(z)$ and $R(z)$ having no common root on $\mathcal{C}$. Let $\underline{\phi} := \min_{z \in \mathcal{C}} |Q(z)/R(z)| = 0$. Then there exists $\bar{\phi} > 0$ such that, under the rule $i_t = \phi \mathbb{E}_t \{v_{t+h}\}$ with $(\phi, h) \in \mathbb{R} \times \mathbb{Z}$, Points (c)(i), (c)(ii), (d)(i), and (d)(ii) of Proposition 5 and Points (b1)-(b2) of Proposition 8 still hold. In addition, if $q_{\mathcal{C}} = 1$ and $Q(1) = 0$, then Points (e1)-(e6) of Proposition 9 still hold.*

Proof: See Appendix A.14. ■

Proposition 10 is essentially a mix of Propositions 8 and 9. It is partly summarized in diagrammatic form in Figure 9, which is a mix of the top of Figure 5 and the bottom of Figure 7. In Subsection 4.2, moving from $r_{\mathcal{C}} = 0$ to $r_{\mathcal{C}} \geq 1$ (and keeping $q_{\mathcal{C}} = 0$) affected the properties of the system for $|\phi|$ higher than $\bar{\phi}$ or in the neighborhood of $\bar{\phi}$, not those for $|\phi|$ below $\underline{\phi}$ or in the neighborhood of $\underline{\phi}$. In Subsection 4.3, on the contrary, moving from $q_{\mathcal{C}} = 0$ to $q_{\mathcal{C}} \geq 1$ (and keeping $r_{\mathcal{C}} = 0$) affected the latter properties, not the former. In the current subsection, moving jointly from $q_{\mathcal{C}} = r_{\mathcal{C}} = 0$ to $q_{\mathcal{C}} \geq 1$ and $r_{\mathcal{C}} \geq 1$ affects both types of properties.

Proposition 10 is illustrated in the bottom right panel of Figure 8 and the two panels of Figure

Figure 9: Determinacy status for systems with $q_{\mathcal{C}} \geq 1$ and $r_{\mathcal{C}} \geq 1$

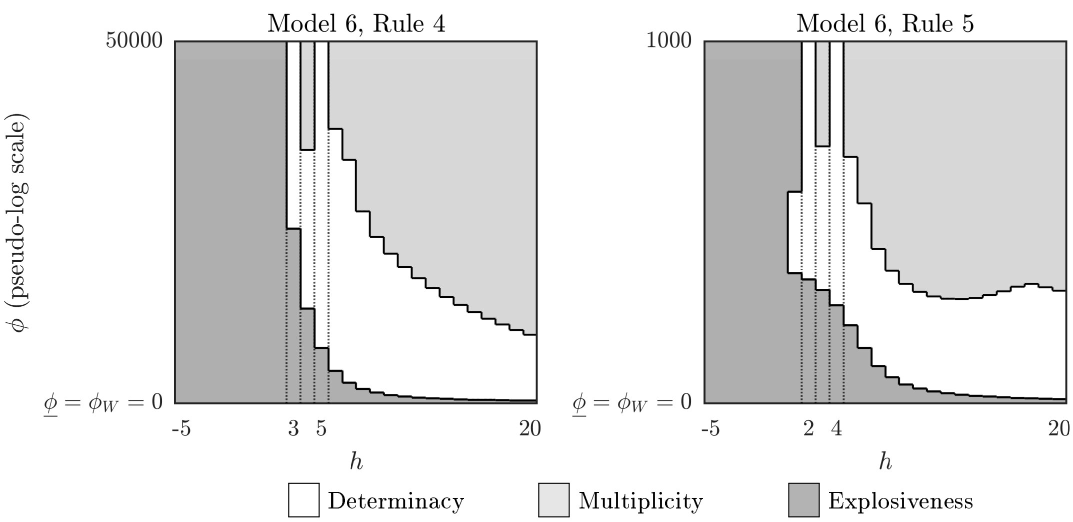


10. The latter figure shows the determinacy status for the systems (Model 6, Rule 4) and (Model 6, Rule 5), again with a pseudo-log scale for ϕ.

In these three panels, we have $r_{\mathcal{C}} \geq 2$, so Point (b1) of Proposition 8 applies, but not Point (b2). More specifically, in the bottom right panel of Figure 8, we have $r_{\mathcal{C}} = 2$ and we get no horizon at which determinacy obtains for a sufficiently high coefficient. In each panel of Figure 10, we have $r_{\mathcal{C}} = 3$ and we get two such horizons.

In these three panels, we also have $q_{\mathcal{C}} = 1$ and $Q(1) = 0$, so Points (e1)-(e6) of Proposition 9 apply. More specifically, these panels illustrate Point (e4) of Proposition 9: we have $q \geq \delta + 1$, so we get explosiveness for ϕ just above 0 ($S^+ = E$). We also get explosiveness for ϕ just below 0 ($S^- = E$, figure not shown).

Figure 10: Determinacy status for Model 6 and Rules 4-5



5 Extensions and discussion

In this section, I extend the results of the previous sections to rules involving several variables and to inertial rules, and I discuss some possible applications of the results of the paper.

5.1 Extension to rules with several variables

I start with rules involving several variables. The rules I have considered so far were of type (5). This type of rule involves a single variable v_t, associated with a single coefficient ϕ and a single horizon h (even though v_t can itself be defined as a linear combination of several variables). I now consider, in addition to v_t, some variables $(v_{1,t}, ..., v_{J,t})$ associated with some coefficients $(\phi_1, ..., \phi_J) \in (\mathbb{R} \setminus \{0\})^J$ and some horizons $(h_1, ..., h_J) \in \mathbb{Z}^J$, where $J \in \mathbb{N} \setminus \{0\}$. I show how to address the same questions as previously, about the determinacy status, the determinacy horizons, and the Taylor principle, still conditionally on the values of ϕ and h, but this time for the rule

$$i_t = \phi \mathbb{E}_t \{v_{t+h}\} + \sum_{j=1}^{J} \phi_j \mathbb{E}_t \left\{v_{j,t+h_j}\right\} \tag{7}$$

instead of Rule (5). To do that, one can simply move the new term $\sum_{j=1}^{J} \phi_j \mathbb{E}_t\{v_{j,t+h_j}\}$ out of the rule and into the structural equations, and then apply the previous results to the resulting "structural equations" and Rule (5). The outcome will, of course, partly depend on the new "structural equations." In general, these equations may not lead to the same determinacy status under a policy-instrument peg as the original structural equations. If $(\phi_1, ..., \phi_J)$ are sufficiently small, however, they will, and the values of $\underline{\phi}$ and $\bar{\phi}$ under Rule (7) will not be far away from their values under Rule (5). In the following, I will be more specific, quantitatively speaking, about what I mean by "sufficiently small" and "not far away."

For any $j \in \{1, ..., J\}$, let m_j, $R_j(z)$, and $\underline{\phi}_j$ denote the counterparts of m, $R(z)$, and $\underline{\phi}$ for variable $v_{j,t}$ instead of variable v_t. Let $g := \max[0, \max_{j\in\{1,...,J\}}(h_j - m_j)]$. I obtain the following results:

Proposition 11 (Determinacy status, determinacy horizons, and Taylor principle under a rule with several variables): *Consider a model (4), some variables $(v_t, v_{1,t}, ..., v_{J,t})$, some coefficients $(\phi, \phi_1, ..., \phi_J) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})^J$, and some horizons $(h, h_1, ..., h_J) \in \mathbb{Z}^{J+1}$, where $J \in \mathbb{N} \setminus \{0\}$. Then, Propositions 5-10 still hold for Rule (7) instead of Rule (5), if δ, m, and $Q(z)$ are respectively replaced by $\delta + g$, $m + g$, and $z^g[Q(z) + \sum_{j=1}^{J} \phi_j R_j(z) z^{m_j - h_j}]$ in these propositions. In addition, if the system composed of Model (4) and Rule (5) is regular and $\sum_{j=1}^{J} |\phi_j| / \underline{\phi}_j < 1$, then, as we move from Rule (5) to Rule (7):*
(a) the system remains regular and the determinacy status for $\phi = 0$ is unchanged;
(b) $\underline{\phi}$ is multiplied by a factor not lower than $1 - \sum_{j=1}^{J} |\phi_j| / \underline{\phi}_j$;
(c) $\bar{\phi}$ is multiplied by a factor not higher than $1 + \sum_{j=1}^{J} |\phi_j| / \underline{\phi}_j$.

Proof: See Appendix A.15. ■

The condition $\sum_{j=1}^{J} |\phi_j| / \underline{\phi}_j < 1$, in this proposition, characterizes the size of a neighborhood of $(\phi_1, ..., \phi_J) = (0, ..., 0)$ within which the determinacy-status results are similar under Rule (5) and under Rule (7), in the sense that these results are described by the same panel of Figure 3 under the two rules (Point (a) of Proposition 11). Under this condition, moreover, the values of $\underline{\phi}$ and $\bar{\phi}$ under Rule (7) lie in a neighborhood of their values under Rule (5); the size of this neighborhood is specified in Points (b)-(c) of Proposition 11.

5.2 Extension to inertial rules

I now turn to inertial rules, i.e. rules making the current policy instrument react to its past values in addition to a variable at horizon h with coefficient ϕ:

$$\rho(L) i_t = \phi \mathbb{E}_t \{v_{t+h}\}, \tag{8}$$

where $\phi \in \mathbb{R}$, $h \in \mathbb{Z}$, and $\rho(z) \in \mathbb{R}[z]$ with $\rho(0) \neq 0$. I obtain the following proposition:

Proposition 12 (Determinacy status, determinacy horizons, and Taylor principle under an inertial rule): *Consider a model (4) and a variable v_t. Then, Propositions 5-10 still hold for Rule (8) instead of Rule (5) and for $\phi \neq 0$, if $Q(z)$ is replaced by $Q(z)\rho(z)$ in these propositions.*

Proof: See Appendix A.16. ■

In essence, this proposition says that adding some inertia to a rule (i.e. replacing i_t by $\rho(L)i_t$ in the rule) simply amounts to adding the same inertia to the dynamic system under a policy-instrument peg (i.e. replacing $Q(z)$ by $Q(z)\rho(z)$).

Consider, for instance, a regular system with a non-inertial rule, and add some inertia to the rule (i.e. replace i_t by $\rho(L)i_t$ in the rule). I distinguish between three alternative cases. First, $\rho(z)$ may have all its roots outside $\mathcal{C}$. In this case, the new system is also regular, $q - \delta$ does not change, and the determinacy status under the new rule is diagrammatically represented by the same panel of Figure 3 as the determinacy status under the original rule. Second, $\rho(z)$ may have at least one root inside $\mathcal{C}$ (i.e. the new rule may be "superinertial" in the sense of Woodford, 2003, Chapter 8, and Giannoni and Woodford, 2002), while still having no root on $\mathcal{C}$. In that case, $q - \delta$ increases by the number of inside roots of $\rho(z)$, and we may move from the left panel of Figure 3 to its middle or right panel, or from its middle panel to its right panel. Third, $\rho(z)$ may have at least one root on $\mathcal{C}$. In that case, $q_{\mathcal{C}}$ increases by the number of roots of $\rho(z)$ on $\mathcal{C}$; so, the new system is non-regular, and we move from a panel of Figure 3 to Figure 7. In particular, when $\rho(z) = 1 - z$ (i.e. when the new rule is a "first-difference rule"), we get $q_{\mathcal{C}} = 1$ and $Q(1) = 0$; so, Points (e1)-(e6) of Proposition 9 apply.

Proposition 12 can be used to design a rule that makes the set of determinacy horizons H_D unbounded below and above in models with $q \leq \delta - 1$. In these models, under the non-inertial rule (5), H_D is bounded above (as shown in Point (b)(i) of Proposition 6) and may be bounded below (as shown in Point (b)(ii) of Proposition 6) if the system is regular. To enlarge H_D (for instance, in order to ensure determinacy in the presence of inside lags), the policymaker can adopt a superinertial rule (8) with exactly $\delta - q$ roots of $\rho(z)$ inside $\mathcal{C}$. Replacing Rule (5) with this rule moves us from the left to the middle panel of Figure 3, and makes H_D unbounded below and above. Thus, with a degree of superinertia equal to the dimension of multiplicity under a peg ($\delta - q$) and with a sufficiently small coefficient ϕ in absolute value, Rule (8) ensures determinacy for any horizon h. This result echoes, and sheds light on, a result obtained by Woodford (2003, Chapter 8) and Giannoni and Woodford (2002, 2003, 2005) about the degree of superinertia of their "robustly optimal rules," which they find is equal to the degree of multiplicity under a peg.

Alternatively, for models with $q \geq \delta$, Proposition 12 implies that any superinertial rule with a sufficiently small coefficient ϕ in absolute value will necessarily lead to explosiveness (again, if the system is regular). In effect, replacing the non-inertial rule (5) with a superinertial rule (8) moves us from the middle to the right panel of Figure 3 (for a model with $q = \delta$), or keeps us in the right panel of Figure 3 (for a model with $q \geq \delta + 1$). This result offers an explanation for the propensity of superinertial rules to generate explosiveness in backward-looking models (Rudebusch and Svensson, 1999, and Levin and Williams, 2003).

5.3 Discussion about applications

I now briefly discuss four different ways to apply the results I have established in this paper.

First, the results can be used to shed light on the determinacy implications, in a broad class of models, of a given specific policy. Consider, for example, the average-inflation-targeting strategy adopted by the Federal Reserve in 2020. There are at least two different ways to interpret and formalize this strategy (see, e.g., Arias et al., 2020, and Mester, 2021). One way is to consider a rule making the interest rate react to the average of inflation over several dates, e.g. Rule (5) with $v_t = \sum_{k=0}^{K} \pi_{t-k}$, where $K \in \mathbb{N} \setminus \{0\}$ (as long as the zero lower bound on nominal interest rates is not binding, of course). Under this rule, $R(z)$ is a multiple of $\sum_{k=0}^{K} z^k$, so $r_{\mathcal{C}} \geq K \geq 1$; therefore, in any model with $q_{\mathcal{C}} = 0$, the determinacy status, as a function of ϕ and h, is characterized by Proposition 8 and Figure 5.

Another way to interpret and formalize this strategy is to consider a rule making the interest rate react to the price level, e.g. Rule (5) with $v_t = p_t$ (again, as long as the zero lower bound on nominal interest rates is not binding). Under that rule, in any model that involves the inflation rate but not the price level per se, we have $q_{\mathcal{C}} \geq 1$; so, the determinacy status is characterized by Proposition 9 and Figure 7, or by Proposition 10 and Figure 9. Thus, the determinacy implications of these two alternative rules – the one involving average inflation, and the other involving the price level – are qualitatively different, at least for $|\phi|$ sufficiently small or large. As Proposition 12 implies, moreover, this conclusion is unchanged when the two rules are inertial, i.e. when they are of type (8) instead of (5), provided that $\rho(z)$ has no roots on $\mathcal{C}$.

Second, my general results provide guidelines for finding rules with robust determinacy properties across alternative models. For example, if a variable v_t leads to $r_{\mathcal{C}} = 0$ in all the models considered and to the same h^* across these models, then Rule (5) with $h = h^*$ and $|\phi|$ sufficiently large will deliver determinacy in all the models (Propositions 5 and 9). As another example, a Wicksellian rule with a sufficiently small positive coefficient ϕ on the price level, regardless of the horizon h, will deliver determinacy in all the models that: (i) involve the inflation rate but not the price level per se, and (ii) are such that ($q = \delta - 1$ and $Q'(1)R(1) > 0$) or ($q = \delta$ and $Q'(1)R(1) < 0$) (Propositions 9-10). As still another example, if all the models considered share the same negative value of $q - \delta$ and are such that $q_{\mathcal{C}} = 0$, then an inertial rule (8) with exactly $\delta - q$ roots of $\rho(z)$ inside $\mathcal{C}$ and with $|\phi|$ sufficiently small will deliver determinacy in all the models (Proposition 12).

Research on the robustness of interest-rate rules across alternative monetary-policy models has, over the past ten years, benefited from the development of a Macroeconomic Model Data Base (MMB) described in Wieland et al. (2012, 2016). The MMB Team (2022) writes that "there is no hard guideline for determinacy" and refers to Levin et al. (2003) for a suggestion of several characteristics of rules that deliver determinacy. Among these characteristics, which Levin et al. (2003) identify numerically using five calibrated models, are "a relatively short inflation forecast horizon" and "a moderate degree of responsiveness to the inflation forecast."

Since the five models considered in Levin et al. (2003) are such that $q \leq \delta - 1$, Propositions 5 and 12 offer an explanation of these two characteristics: for $h \geq h^* + 1$, two necessary conditions for determinacy are that h should be sufficiently small and that $|\phi|$ should be between $\underline{\phi}$ and $\bar{\phi}$, no matter whether the rule is non-inertial or inertial (provided that $\rho(z)$ has no roots inside or on $\mathcal{C}$). Moreover, these propositions show that these two characteristics, qualitatively speaking, remain necessary for determinacy for a broad range of model calibrations (not just the ones considered in Levin et al., 2003), a broad class of models (not just the five models they consider), and a broad class of variables in the rule (not just inflation) – in essence, for all the calibrated models such that $q \leq \delta - 1$, and all the variables v_t that make the system regular.

Like the five models considered in Levin et al. (2003), most of the models in the current MMB version (3.1) are such that $q \leq \delta - 1$, i.e. most of them deliver multiplicity under an interest-rate peg. Table 4 reports the distribution of $q - \delta$ across the 140 rational-expectations models in the base: 90% of these models are, more specifically, such that $q = \delta - 1$. Computing the thresholds $\underline{\phi}$, $\bar{\phi}$, h^*, and h^{**} for various models in the base and various variables in the rule could be helpful in the quest for a robust rule.

Table 4: Distribution of $q-\delta$ in the Macroeconomic Model Data Base 3.1

Value of $q-\delta$	-2	-1	0	1
Number of models	4	126	4	6

Third, although I have illustrated my results only with monetary-policy models, these results can be applied to any stabilization policy, like fiscal policy for instance. Fiscal-policy models may, however, raise a specific difficulty related to Blanchard and Kahn's (1980) no-decoupling condition, which I discuss in Loisel (2021a). More specifically, when the tax-rate rule does not involve the debt level, the log-linearized dynamic system may have a block-recursive structure in which the debt level is residually determined and explodes over time (at a rate equal to the steady-state real interest rate), while we may get determinacy or multiplicity for the other variables. This difficulty disappears, i.e. the no-decoupling condition is met, as soon as the tax-rate rule involves the debt level. So, my results can be applied to such models as long as the variable v_t in the rule is the debt level, and the coefficient ϕ on this variable is non-zero.

Fourth and last, I have so far considered the variable i_t as the policy instrument, and focused on policy-instrument rules. However, nothing prevents us from interpreting i_t as yet another endogenous variable set by the private sector, instead of a policy instrument. So, the results obtained could be applied to "targeting rules," instead of policy-instrument rules. They could also be applied to structural equations (which describe the behavior of the private sector), instead of targeting rules or policy-instrument rules (which describe the behavior of a policymaker). For instance, in a monetary-policy model, the results could be used to find conditions on the structural parameters for the model to deliver determinacy under an interest-rate peg, in order to solve New Keynesian puzzles and paradoxes at the zero lower bound.

6 Conclusion

This paper has established some simple, necessary or sufficient conditions for determinacy in a broad class of dynamic rational-expectations models that arguably includes, in particular, most existing dynamic stochastic general-equilibrium (DSGE) models. These determinacy conditions are directly about the coefficients and horizons of the policy-instrument rule, and lead to new, general principles for stabilization policy. In so doing, the paper has provided new insights into why a given rule does or does not deliver determinacy in a given model; it has shed light on various determinacy or indeterminacy results obtained in the literature and sparsely distributed across models and rules; and it has provided some first hard guidelines for finding rules with robust determinacy properties across alternative models. The results can be applied to monetary policy, fiscal policy, or any other stabilization policy; I have used them, for instance, to highlight the different determinacy implications, in a broad class of models, of different ways to implement the average-inflation-targeting strategy adopted by the Federal Reserve in 2020. Overall, the paper thus opens new horizons for the study of stabilization policy, and paves the way for new qualitative or quantitative research.

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Appendix

A.1 Proof of Points (c)-(d) of Proposition 1

Point (c). For $h \geq 2$, we have $\nu = h$ and

$$P(z) = Q(z)z^{h-2} + \frac{\phi\kappa}{\sigma}.$$

Let z_o denote the root of $Q(z)$ in $(1, +\infty)$, with the subscript "o" standing for "outside $\mathcal{C}$." Consider a Jordan curve $\mathcal{J}_o$ surrounding z_o and not intersecting nor surrounding $\mathcal{C}$. I apply Rouché's theorem to $\mathcal{J} = \mathcal{J}_o$, $P_b(z) = Q(z)z^{h-2}$, and $P_s(z) = \phi\kappa/\sigma$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \geq \bar{h} := 2 + \max\left\{0, \left\lceil \frac{\log\left(\frac{\bar{\phi}\kappa}{\sigma}\right) - \log\left(\min_{\tilde{z}\in\mathcal{J}_o} |Q(\tilde{z})|\right)}{\log\left(\min_{\tilde{z}\in\mathcal{J}_o} |\tilde{z}|\right)} \right\rceil\right\},$$

and any $z \in \mathcal{J}_o$, we have

$$\left|Q(z)z^{h-2}\right| \geq \min_{\tilde{z}\in\mathcal{J}_o} \left|Q(\tilde{z})\,\tilde{z}^{h-2}\right| \geq \left(\min_{\tilde{z}\in\mathcal{J}_o} |Q(\tilde{z})|\right)\left(\min_{\tilde{z}\in\mathcal{J}_o} |\tilde{z}|\right)^{h-2} \geq \frac{\bar{\phi}\kappa}{\sigma} > \left|\frac{\phi\kappa}{\sigma}\right|,$$

where the last but one inequality follows from the definition of $\bar{h}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{J}_o$ as $Q(z)z^{h-2}$. The latter polynomial has exactly one root inside $\mathcal{J}_o$, which is z_o. Therefore, $P(z)$ has also exactly one root inside $\mathcal{J}_o$, and hence at least one root outside $\mathcal{C}$. Since the degree of $P(z)$ is h, we thus get $p \leq h-1 < h = \nu$, and consequently $S(\phi, h) = M$ for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \geq \bar{h}$.

Point (d). For $h \leq 2$, we have $\nu = 2$ and

$$P(z) = Q(z) + \frac{\phi\kappa}{\sigma} z^{2-h}.$$

I proceed in four steps. In the first step, I show that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, all but one root of $P(z)$ converge uniformly to $\mathcal{C}$ as $h \to -\infty$. I get this result by applying Rouché's theorem twice. Consider an arbitrary $\epsilon \in (0, 1 - z_i)$, where z_i denotes the root of $Q(z)$ in $(0,1)$, with the subscript "i" standing for "inside $\mathcal{C}$." For any $r \in \mathbb{R}_+$, let $\mathcal{C}_r$ denote the circle of radius r centered at the origin of the complex plane (so that in particular $\mathcal{C}_1 = \mathcal{C}$). I first apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1-\epsilon}$, $P_b(z) = Q(z)$, and $P_s(z) = (\phi\kappa/\sigma)z^{2-h}$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \leq \underline{h}_{1-\epsilon} := 2 + \min\left\{0, \left\lfloor \frac{\log\left(\min_{\tilde{z}\in\mathcal{C}_{1-\epsilon}} |Q(\tilde{z})|\right) - \log\left(\frac{\bar{\phi}\kappa}{\sigma}\right)}{-\log(1-\epsilon)} \right\rfloor\right\},$$

and any $z \in \mathcal{C}_{1-\epsilon}$, we have

$$|Q(z)| \geq \min_{\tilde{z}\in\mathcal{C}_{1-\epsilon}} |Q(\tilde{z})| \geq \frac{\bar{\phi}\kappa}{\sigma}(1-\epsilon)^{2-h} > \frac{|\phi|\,\kappa}{\sigma}(1-\epsilon)^{2-h} = \left|\frac{\phi\kappa}{\sigma} z^{2-h}\right|,$$

where the second inequality follows from the definition of $\underline{h}_{1-\epsilon}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}_{1-\epsilon}$ as $Q(z)$. The latter polynomial has exactly one root inside $\mathcal{C}_{1-\epsilon}$, which is z_i. Therefore, $P(z)$ has also exactly one root inside $\mathcal{C}_{1-\epsilon}$ for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \leq \underline{h}_{1-\epsilon}$.

I then apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1+\epsilon}$, $P_b(z) = (\phi\kappa/\sigma)z^{2-h}$, and $P_s(z) = Q(z)$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \leq \underline{h}_{1+\epsilon} := 2 + \min\left\{0, \left\lfloor \frac{\log\left(\frac{\phi\kappa}{\sigma}\right) - \log\left(\max_{\tilde{z}\in\mathcal{C}_{1+\epsilon}} |Q(\tilde{z})|\right)}{\log(1+\epsilon)} \right\rfloor\right\},$$

and any $z \in \mathcal{C}_{1+\epsilon}$, we have

$$\left|\frac{\phi\kappa}{\sigma} z^{2-h}\right| = \frac{|\phi|\,\kappa}{\sigma}(1+\epsilon)^{2-h} > \frac{\phi\kappa}{\sigma}(1+\epsilon)^{2-h} \geq \max_{\tilde{z}\in\mathcal{C}_{1+\epsilon}} |Q(\tilde{z})| \geq |Q(z)|,$$

where the last but one inequality follows from the definition of $\underline{h}_{1+\epsilon}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}_{1+\epsilon}$ as $(\phi\kappa/\sigma)z^{2-h}$. Therefore, $P(z)$ has exactly $2-h$ roots inside $\mathcal{C}_{1+\epsilon}$ for any $h \leq \underline{h}_{1+\epsilon}$. Since the degree of $P(z)$ is $2-h$ when $h \leq 0$, we eventually get that for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \leq \min\left(0, \underline{h}_{1-\epsilon}, \underline{h}_{1+\epsilon}\right)$, all but one root of $P(z)$ lie between $\mathcal{C}_{1-\epsilon}$ and $\mathcal{C}_{1+\epsilon}$. We conclude that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, all but one root of $P(z)$ converge uniformly to $\mathcal{C}$ as $h \to -\infty$.

In the second step, I show that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, the roots of $P(z)$ uniformly converging to $\mathcal{C}$ as $h \to -\infty$ converge in distribution to the uniform distribution on $\mathcal{C}$. This result is a direct consequence of the following theorem:

Theorem 2 (Erdős and Turán, 1950): *Let* $\tilde{P}(z) = \sum_{k=0}^{d} \tilde{p}_k z^k \in \mathbb{C}[z]$ *with* $\tilde{p}_0\tilde{p}_d \neq 0$. *Let* $\varphi_k \in [0, 2\pi)$ *for* $1 \leq k \leq d$ *denote the angular coordinates of the roots of* $\tilde{P}(z)$. *For any* $0 \leq \underline{\alpha} < \bar{\alpha} \leq 2\pi$,

$$\left|\#\left\{k \in \{1,...,d\} | \underline{\alpha} \leq \varphi_k < \bar{\alpha}\right\} - \left(\frac{\bar{\alpha} - \underline{\alpha}}{2\pi}\right) d\right| \leq 16\sqrt{d \log\left(\frac{1}{\sqrt{|\tilde{p}_0\tilde{p}_d|}} \sum_{k=0}^{d} |\tilde{p}_k|\right)}.$$

Proof: See Erdős and Turán (1950). ■

I apply this theorem to $\tilde{P}(z) = P(z)$. For $\tilde{P}(z) = P(z)$ and $h \leq -1$, we have

$$\frac{1}{\sqrt{|\tilde{p}_0\tilde{p}_d|}} \sum_{k=0}^{d} |\tilde{p}_k| = 2(1+\beta) + \frac{\kappa}{\sigma}(1+\phi).$$

So, the Erdős-Turán theorem straightforwardly implies, together with the result of the previous step, that all but one root of $P(z)$ uniformly converge in distribution to the uniform distribution on $\mathcal{C}$ as $h \to -\infty$, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$.

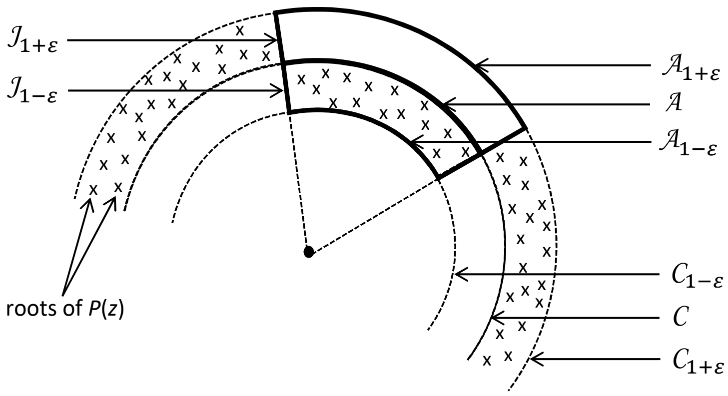
In the third step, I show that the number of roots of $P(z)$ inside $\mathcal{C}$ grows unboundedly as $h \to -\infty$, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$. Since $|\phi| > \underline{\phi}$, there exists an arc $\mathcal{A}$ of $\mathcal{C}$ such that $\forall z \in \mathcal{A}$,

$|\phi|\,\kappa/\sigma > |Q(z)|$. For any $r \in \mathbb{R}_+$, let $\mathcal{A}_r$ denote the image of $\mathcal{A}$ under the homothety whose center is the origin of the complex plane and whose ratio is r (so that in particular $\mathcal{A}_1 = \mathcal{A}$). By continuity, there exists $\varepsilon \in (0,1)$ such that $|\phi|\,\kappa/\sigma > |Q(z)|$ for all z on the Jordan curve $\mathcal{J}_{1+\varepsilon}$ made of $\mathcal{A}$, $\mathcal{A}_{1+\varepsilon}$, and the two radial line segments joining the endpoints of $\mathcal{A}$ and $\mathcal{A}_{1+\varepsilon}$ (see Figure A.1). I apply Rouché's theorem to $\mathcal{J} = \mathcal{J}_{1+\varepsilon}$, $P_b(z) = (\phi\kappa/\sigma)z^{2-h}$, and $P_s(z) = Q(z)$. For any $h \leq 2$ and any $z \in \mathcal{J}_{1+\varepsilon}$, we have

$$\left|\frac{\phi\kappa}{\sigma}z^{2-h}\right| \geq \frac{|\phi|\,\kappa}{\sigma} > |Q(z)|\,.$$

So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{J}_{1+\varepsilon}$ as $(\phi\kappa/\sigma)z^{2-h}$. Therefore, $P(z)$ has no roots inside $\mathcal{J}_{1+\varepsilon}$ for any $h \leq 2$. Using the results of the first two steps, we get that the number of roots of $P(z)$ inside the Jordan curve $\mathcal{J}_{1-\varepsilon}$ made of $\mathcal{A}_{1-\varepsilon}$, $\mathcal{A}$, and the two radial line segments joining the endpoints of $\mathcal{A}_{1-\varepsilon}$ and $\mathcal{A}$, grows unboundedly as $h \to -\infty$. As a result, p grows unboundedly as $h \to -\infty$. Thus, there exists $\underline{h}\,(|\phi|)$ such that $p > 2 = \nu$ and $S(\phi, h) = E$ for all $h \leq \underline{h}\,(|\phi|)$.

Figure A.1: Roots of $P(z)$ as $h \to -\infty$



In the fourth step, I just note that for any $\varepsilon \in \left(0, \bar{\phi} - \underline{\phi}\right)$ and any $|\phi| \in (\underline{\phi} + \varepsilon, \bar{\phi})$, there exists, by continuity, $\ell(\varepsilon) > 0$ such that the arc $\mathcal{A}$ can be chosen of length higher than $\ell(\varepsilon)$. As a result, $\underline{h}\,(|\phi|)$ can be chosen a bounded function of $|\phi|$ for $|\phi| \in (\underline{\phi} + \varepsilon, \bar{\phi})$.

A.2 Determination of $\underline{\phi}$ and $\bar{\phi}$ in the basic New Keyn. model under Rule 1

For any $z \in \mathcal{C}$, we have

$$|Q(z)| = \left|\beta - \left(1 + \beta + \frac{\kappa}{\sigma}\right)z + z^2\right| \leq \beta + \left(1 + \beta + \frac{\kappa}{\sigma}\right)|z| + |z|^2 = 2\,(1+\beta) + \frac{\kappa}{\sigma},$$

with equality only for $z = -1$. Therefore, $\arg\max_{z\in\mathcal{C}} |Q(z)| = \{-1\}$, $\max_{z\in\mathcal{C}} |Q(z)| = 2\,(1+\beta) + \kappa/\sigma$, and

$$\bar{\phi} := \frac{\sigma}{\kappa}\max_{z\in\mathcal{C}} |Q(z)| = 1 + 2\,(1+\beta)\,\frac{\sigma}{\kappa}.$$

For any $z = a + ib \in \mathcal{C}$, where $(a, b) \in [-1, 1]^2$ and $a^2 + b^2 = 1$, some simple algebra leads to $|Q(z)|^2 = T_1(a)$, where

$$T_1(a) := 4\beta a^2 - 2(1+\beta)\left(1 + \beta + \frac{\kappa}{\sigma}\right) a + \left[(1-\beta)^2 + \left(1 + \beta + \frac{\kappa}{\sigma}\right)^2\right].$$

For any $a \in [-1, 1]$, we have $T_1'(a) \leq T_1'(1) = -2(1-\beta)^2 - 2(1+\beta)\kappa/\sigma < 0$. So, $T_1(a)$ is decreasing in a over $[-1, 1]$. Therefore, $\arg\min_{a\in[-1,1]} T_1(a) = \{1\}$, $\arg\min_{z\in\mathcal{C}} |Q(z)| = \{1\}$, $\min_{z\in\mathcal{C}} |Q(z)| = \kappa/\sigma$, and

$$\underline{\phi} := \frac{\sigma}{\kappa} \min_{z\in\mathcal{C}} |Q(z)| = 1.$$

A.3 Determination of $\underline{\phi}$ and $\bar{\phi}$ in the basic New Keyn. model under Rule 2

For any $z = a + ib \in \mathcal{C}$, where $(a, b) \in [-1, 1]^2$ and $a^2 + b^2 = 1$, some simple algebra leads to $|Q(z)/(z-\beta)|^2 = T_2(a) := T_1(a)/(1 + \beta^2 - 2\beta a)$, where $T_1(a)$ is defined in Appendix A.2. We have $T_2'(a) = T_3(a)/(1 + \beta^2 - 2\beta a)^2$, where

$$T_3(a) := -8\beta^2 a^2 + 8\beta\left(1 + \beta^2\right) a + 2\eta$$

with

$$\eta := \left[(1-\beta)^2 + \left(1 + \beta + \frac{\kappa}{\sigma}\right)^2\right]\beta - (1+\beta)\left(1+\beta^2\right)\left(1 + \beta + \frac{\kappa}{\sigma}\right).$$

For any $a \in [-1, 1]$, we have $T_3'(a) \geq T_3'(1) = 8\beta(1-\beta)^2 > 0$. So, $T_3(a)$ is increasing in a over $[-1, 1]$. There are, therefore, three possible alternative cases: (i) $T_3(-1) > 0$, (ii) $T_3(-1) < 0 < T_3(1)$, and (iii) $T_3(1) < 0$. In Case (i), $T_2(a)$ is increasing in a over $[-1, 1]$; in Case (ii), $T_2(a)$ is first decreasing and then increasing in a over $[-1, 1]$; and in Case (iii), $T_2(a)$ is decreasing in a over $[-1, 1]$. In all three cases, $\arg\max_{a\in[-1,1]} T_2(a) \subset \{-1, 1\}$, hence $\arg\max_{z\in\mathcal{C}} |Q(z)/(z-\beta)| \subset \{-1, 1\}$, and therefore

$$\bar{\phi} := \sigma \max_{z\in\mathcal{C}} \left|\frac{Q(z)}{z-\beta}\right| = \max\left(\frac{\kappa}{1-\beta}, 2\sigma + \frac{\kappa}{1+\beta}\right).$$

The double inequality $T_3(-1) < 0 < T_3(1)$ is equivalent to

$$\left|\eta - 4\beta^2\right| < 4\beta\left(1 + \beta^2\right). \tag{A.1}$$

If Condition (A.1) is not met, then we are in Case (i) or (iii), so $\arg\min_{a\in[-1,1]} T_2(a) \subset \{-1, 1\}$, and therefore $\arg\min_{z\in\mathcal{C}} |Q(z)/(z-\beta)| \subset \{-1, 1\}$. Alternatively, if Condition (A.1) is met, then we are in Case (ii), so $\arg\min_{a\in[-1,1]} T_2(a) = \{a^*\}$, where $a^* := [(1+\beta^2) - \sqrt{(1+\beta^2)^2 + \eta}]/(2\beta)$ is the root of $T_3(a)$ in $[-1, 1]$, and therefore $\arg\min_{z\in\mathcal{C}} |Q(z)/(z-\beta)| = \{a^* - i\sqrt{1 - a^{*2}}, a^* + i\sqrt{1 - a^{*2}}\}$. As a consequence,

$$(A.1) \implies \underline{\phi} := \sigma \min_{z\in\mathcal{C}} \left|\frac{Q(z)}{z-\beta}\right| = \frac{\sigma}{\sqrt{\beta}}\sqrt{(1+\beta)\frac{\kappa}{\sigma} - (1-\beta)^2 + 2\sqrt{(1+\beta^2)^2 + \eta}},$$

$$\neg(A.1) \implies \underline{\phi} := \sigma \min_{z\in\mathcal{C}} \left|\frac{Q(z)}{z-\beta}\right| = \min\left(\frac{\kappa}{1-\beta}, 2\sigma + \frac{\kappa}{1+\beta}\right).$$

A.4 Proof of Proposition 3

The equality $\phi_W = \underline{\phi}$ stated in Proposition 3 straightforwardly follows from $\phi_W := 1$ (as explained in the main text) and $\underline{\phi} = 1$ (as shown in Appendix A.2).

Points (b) and (d). Point (b) of Proposition 3 straightforwardly follows from the definition of H_D. The "if" part of Point (d) of Proposition 3 is a very well known result whose proof can be found in, e.g., Woodford (2003, Chapter 4). The "only if" part of Point (d) of Proposition 3 straightforwardly follows from Point (b) of Proposition 1.

Point (c). I start by rewriting $P(z)$ as a function of two variables: $\hat{P}(\phi, z) := Q(z)z^{\max(0,h-2)} + (\phi\kappa/\sigma)z^{\max(0,2-h)}$, where $(\phi, z) \in \mathbb{R} \times \mathbb{C}$. Simple algebra leads to $\hat{P}(1,1) = 0$ and $\partial\hat{P}/\partial z(1,1) = 1-\beta+(1-h)\kappa/\sigma$. This last expression is generically non-zero (it can be zero only if $(1-\beta)\sigma/\kappa$ is an integer, and I ignore this zero-measure case). So, one root of the polynomial $\hat{P}(1,z)$ is 1, and this root is of multiplicity one. The implicit-function theorem implies the existence of a continuously differentiable function $\phi \mapsto Z(\phi)$ such that one real root of $P(z)$ can be written as $Z(\phi)$ in the neighborhood of $\phi = 1$, with $Z(1) = 1$ and

$$Z'(1) = \frac{-\frac{\partial \hat{P}}{\partial \phi}(1,1)}{\frac{\partial \hat{P}}{\partial z}(1,1)} = \frac{1}{h - \left[1 + (1-\beta)\frac{\sigma}{\kappa}\right]}.$$

For any $h \in \{h \in \mathbb{Z} | h < 1 + (1-\beta)\sigma/\kappa\}$, we have $Z'(1) < 0$, and therefore the root of $P(z)$ goes from outside to inside $\mathcal{C}$ as ϕ crosses 1 from below. It is the only root that crosses $\mathcal{C}$ as ϕ goes through 1. Indeed, any root $z \in \mathbb{C}$ having this property must satisfy $\hat{P}(1,z) = 0$, which implies $|Q(z)| = \kappa/\sigma$ and hence $z = 1$ (since $\min_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})| = \kappa/\sigma$ and $\arg\min_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})| = \{1\}$, as shown in Appendix A.2). So, the number of roots of $P(z)$ inside $\mathcal{C}$ increases by exactly one as ϕ crosses 1 from below. We know from Subsection 2.2 that this number is $p = \max(1, h-1) < \nu$ for ϕ just below $\underline{\phi} = 1 = \phi_W$. Therefore, we have $p = \max(2, h) = \nu$ for ϕ just above $\underline{\phi} = 1 = \phi_W$. As a result, the determinacy status moves from M to D as ϕ crosses 1 from below. Thus, the Taylor principle is locally necessary and sufficient for determinacy for any $h \in \{h \in \mathbb{Z} | h < 1 + (1-\beta)\sigma/\kappa\}$.

Point (a). The result just above straightforwardly implies that $\{h \in \mathbb{Z} | h < 1 + (1-\beta)\sigma/\kappa\} \subset H_D$. To prove the reverse inclusion, I first show that $P'(z)$ has a real root higher than 1 for any $h > 1+(1-\beta)\sigma/\kappa$. If $h = 2 > 1+(1-\beta)\sigma/\kappa$, then $P'(z) = -(1+\beta+\kappa/\sigma)+2z$, the unique root of $P'(z)$ is $(1+\beta+\kappa/\sigma)/2$, and this root is indeed higher than 1. If $h \geq 3$ and $h > 1+(1-\beta)\sigma/\kappa$, then $P'(z) = z^{h-3}T_4(z)$, where $T_4(z) := \beta(h-2) - (1+\beta+\kappa/\sigma)(h-1)z + hz^2$; moreover, we have $T_4(1) = -(\kappa/\sigma)h + 1 - \beta + \kappa/\sigma < 0$ and $\lim_{z\in\mathbb{R}, z\to+\infty} T_4(z) = +\infty$; therefore, $T_4(z)$ has a real root above 1, and so has $P'(z)$. I then use the Gauss-Lucas theorem (first proved by Lucas, 1879, but used earlier by Gauss):

Theorem 3 (Gauss-Lucas theorem): *For any non-constant $\tilde{P}(z) \in \mathbb{C}[z]$, all the roots of $\tilde{P}'(z)$ belong to the convex hull of the set of roots of $\tilde{P}(z)$.*

Proof: See Henrici (1988, Pages 463-464). ■

Applied to $\tilde{P}(z) = P(z)$, this theorem implies that if $P'(z)$ has a real root higher than 1, then $P(z)$ has at least one root outside $\mathcal{C}$. So, for any $h > 1 + (1-\beta)\sigma/\kappa$ and any $\phi \in \mathbb{R}$, $P(z)$ has at least one root outside $\mathcal{C}$, which implies $p < d_P = \max(2, h) = \nu$ (where d_P denotes the degree of $P(z)$), and hence $S(\phi, h) = M$.[1] As a result, $H_D = \{h \in \mathbb{Z} | h < 1 + (1-\beta)\sigma/\kappa\}$.

A.5 Proof of Proposition 4

Points (a) and (c)-(d). Points (a) and (c) of Proposition 4 straightforwardly follow from, respectively, Point (c) of Proposition 2 and the definition of H_D. The "if" part of Point (d) of Proposition 4 is a very well known result whose proof can be found in, e.g., Woodford (2003, Chapter 4). The "only if" part of Point (d) of Proposition 4 straightforwardly follows from Point (b) of Proposition 2.

Point (e). We have $\phi_W := \kappa/(1-\beta)$ and, as shown in Appendix A.3, $\bar{\phi} = \max[\kappa/(1-\beta), 2\sigma + \kappa/(1+\beta)]$. So, we get $\phi_W = \bar{\phi}$ if and only if $\kappa/(1-\beta) \geq 2\sigma + \kappa/(1+\beta)$, that is to say if and only if $\kappa/\sigma \geq (1+\beta)(1-\beta)^2/\beta$. This result corresponds to Point (e)(i) of Proposition 4. Points (e)(ii)-(iv) of Proposition 4 straightforwardly follow from Point (b) of Proposition 2.

Point (f). The condition stated in this point is equivalent to the condition $T_3(1) < 0$ in Appendix A.3. I have shown in Appendix A.3 that $A_{min} := \arg\min_{z \in \mathcal{C}} |Q(z)/(z-\beta)| = \{1\}$ and $\underline{\phi} = \kappa/(1-\beta)$ if this condition is met, and that $1 \notin A_{min}$ and $\underline{\phi} \neq \kappa/(1-\beta)$ if this condition is not met. This result, together with $\phi_W := \kappa/(1-\beta)$, corresponds to Point (f)(i) of Proposition 4.

The proof of Point (f)(ii) of Proposition 4 is similar to the proof of Point (c) of Proposition 3 in Appendix A.4. I rewrite again $P(z)$ as a function of two variables: $\hat{P}(\phi, z) := Q(z)z^{\max(0,h-1)} + (\phi/\sigma)(z-\beta)z^{\max(0,1-h)}$, where $(\phi, z) \in \mathbb{R} \times \mathbb{C}$. Simple algebra leads to $\hat{P}(\phi_W, 1) = 0$ and $\partial\hat{P}/\partial z(\phi_W, 1) = 1 - \beta + [1/(1-\beta) - h]\kappa/\sigma$. This last expression is generically non-zero (it can be zero only if $1/(1-\beta) + (1-\beta)\sigma/\kappa$ is an integer, and I ignore this zero-measure case). So, one root of the polynomial $\hat{P}(\phi_W, z)$ is 1, and this root is of multiplicity one. The implicit-function theorem implies the existence of a continuously differentiable function $\phi \mapsto Z(\phi)$ such that one real root of $P(z)$ can be written as $Z(\phi)$ in the neighborhood of $\phi = \phi_W$, with $Z(\phi_W) = 1$ and

$$Z'(\phi_W) = \frac{-\frac{\partial\hat{P}}{\partial\phi}(\phi_W, 1)}{\frac{\partial\hat{P}}{\partial z}(\phi_W, 1)} = \frac{\frac{1-\beta}{\kappa}}{h - \left[\frac{1}{1-\beta} + (1-\beta)\frac{\sigma}{\kappa}\right]}.$$

This root of $P(z)$ crosses $\mathcal{C}$ at point 1 as ϕ goes through ϕ_W. It is the only root that crosses $\mathcal{C}$ as ϕ goes through ϕ_W. Indeed, any root $z \in \mathbb{C}$ having this property must satisfy $\hat{P}(\phi_W, z) = 0$,

[1] More generally, throughout the Appendix, for any $T(z) \in \mathbb{R}[z]$, d_T denotes the degree of $T(z)$.

which implies $|Q(z)/(z-\beta)| = \kappa/[(1-\beta)\sigma]$ and hence $z = 1$ (since $\min_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})/(\tilde{z}-\beta)| = \kappa/[(1-\beta)\sigma]$ and $\arg\min_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})/(\tilde{z}-\beta)| = \{1\}$, as follows from the analysis in Appendix A.3).

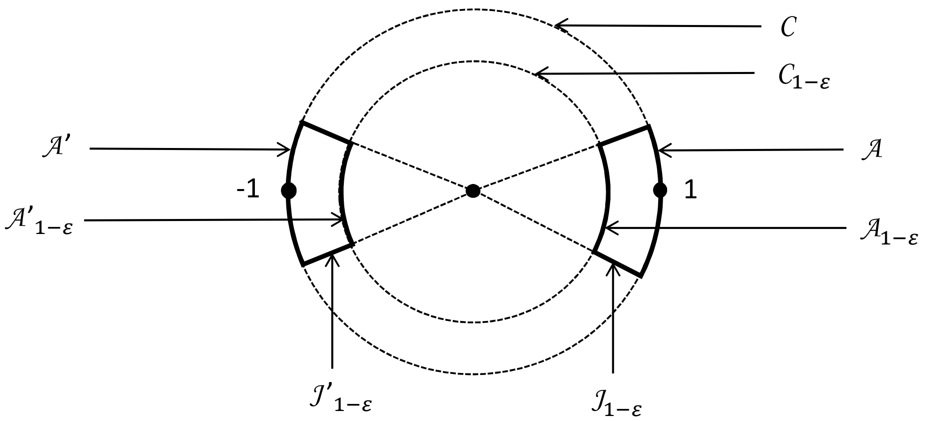
For any $h < 1/(1-\beta) + (1-\beta)\sigma/\kappa$, we have $Z'(\phi_W) < 0$, and therefore the root of $P(z)$ goes from outside to inside $\mathcal{C}$ as ϕ crosses ϕ_W from below. So, the number of roots of $P(z)$ inside $\mathcal{C}$ increases by exactly one, from $p = \max(1, h-1) < \nu$ to $p = \max(2, h) = \nu$, and the determinacy status moves from M to D, as ϕ crosses ϕ_W from below. Thus, the Taylor principle is locally necessary and sufficient for determinacy for any $h < 1/(1-\beta) + (1-\beta)\sigma/\kappa$. Alternatively, for $h > 1/(1-\beta) + (1-\beta)\sigma/\kappa$, we have $Z'(\phi_W) > 0$; as ϕ crosses ϕ_W from below, therefore, the root of $P(z)$ goes this time from inside to outside $\mathcal{C}$, and the determinacy status remains M. Thus, the Taylor principle is not locally necessary and sufficient for determinacy for any $h > 1/(1-\beta) + (1-\beta)\sigma/\kappa$.

Point (b). The condition stated in this point is the same as Condition (A.1) in Appendix A.3. I have shown in Appendix A.3 that $A_{min} \subset \mathcal{C} \setminus \{-1, 1\}$ if this condition is met, and that $A_{min} \subset \{-1, 1\}$ if this condition is not met.

Suppose first that this condition is met, and hence that $A_{min} \subset \mathcal{C} \setminus \{-1, 1\}$. Then, $|Q(z)| > (\underline{\phi}/\sigma)\,|z-\beta|$ for $z \in \{-1, 1\}$. So, by continuity, there exist $\epsilon \in (0, \bar{\phi} - \underline{\phi})$ and two open arcs $\mathcal{A}$ and $\mathcal{A}'$ of $\mathcal{C}$ such that: (i) $1 \in \mathcal{A}$, (ii) $-1 \in \mathcal{A}'$, and (iii) $\forall\,|\phi| \in (\underline{\phi}, \underline{\phi} + \epsilon)$, $\forall z \in \mathcal{A} \cup \mathcal{A}'$, $|Q(z)| > (|\phi|/\sigma)\,|z-\beta|$. For any $r \in \mathbb{R}_+$, let $\mathcal{A}_r$ and $\mathcal{A}'_r$ denote respectively the images of $\mathcal{A}$ and $\mathcal{A}'$ under the homothety whose center is the origin of the complex plane and whose ratio is r (so that in particular $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{A}'_1 = \mathcal{A}'$). In addition, for any $r \in \mathbb{R}_+ \setminus \{0\}$, let $\mathcal{J}_r$ (resp. $\mathcal{J}'_r$) denote the Jordan curve made of $\mathcal{A}$ (resp. $\mathcal{A}'$), $\mathcal{A}_r$ (resp. $\mathcal{A}'_r$), and the two radial line segments joining the endpoints of $\mathcal{A}$ (resp. $\mathcal{A}'$) and $\mathcal{A}_r$ (resp. $\mathcal{A}'_r$). By continuity, there exists $\varepsilon \in (0, 1-z_i)$ such that $\forall\,|\phi| \in (\underline{\phi}, \underline{\phi} + \epsilon)$, $\forall h \leq 1$, $\forall z \in \mathcal{J}_{1-\varepsilon} \cup \mathcal{J}'_{1-\varepsilon}$ (see Figure A.2),

$$|Q(z)| > \frac{|\phi|}{\sigma}\,|z-\beta| \geq \left|\frac{\phi}{\sigma}\,(z-\beta)\,z^{1-h}\right|.$$

Figure A.2: Jordan curves $\mathcal{J}_{1-\varepsilon}$ and $\mathcal{J}'_{1-\varepsilon}$



Applying Rouché's theorem to $P_b(z) = Q(z)$, $P_s(z) = (\phi/\sigma)(z-\beta)z^{1-h}$, and (alternatively)

$\mathcal{J}_{1-\varepsilon}$ and $\mathcal{J}'_{1-\varepsilon}$, I obtain that $\forall\,|\phi| \in (\underline{\phi}, \underline{\phi}+\epsilon)$, $\forall h \leq 1$, $P(z)$ has no root inside $\mathcal{J}_{1-\varepsilon}$ and no root inside $\mathcal{J}'_{1-\varepsilon}$. Therefore, $P(z)$ has no real root between $\mathcal{C}_{1-\varepsilon}$ and $\mathcal{C}$. Now, we know from Appendix A.1 that there exists $\underline{h}_{1-\varepsilon} \in \mathbb{Z}$ such that $\forall\,|\phi| \in (\underline{\phi}, \bar{\phi})$, $\forall h \leq \underline{h}_{1-\varepsilon}$, $P(z)$ has exactly one root inside $\mathcal{C}_{1-\varepsilon}$. As a result, $\forall\,|\phi| \in (\underline{\phi}, \underline{\phi}+\epsilon)$, $\forall h \leq \min(1, \underline{h}_{1-\varepsilon})$, $P(z)$ has exactly one real root inside $\mathcal{C}$, and therefore an odd number p of roots inside $\mathcal{C}$. Since there are $\nu = 2$ non-predetermined variables for any $h \leq 1$, we have $p \neq \nu$, and hence $S(\phi, h) \neq D$, for all $|\phi| \in (\underline{\phi}, \underline{\phi}+\epsilon)$ and all $h \leq \min(1, \underline{h}_{1-\varepsilon})$. Together with Point (d) of Proposition 2, this result implies that H_D is bounded below.

Now suppose alternatively that the condition stated in Point (b) of Proposition 4 is not met, and hence that either $\eta - 4\beta^2 < -4\beta\left(1+\beta^2\right)$, or $\eta - 4\beta^2 > 4\beta\left(1+\beta^2\right)$. In the first case, Point (f) of Proposition 4 implies that H_D is unbounded below. In the second case, we have $A_{min} = \{-1\}$ and $\underline{\phi} = 2\sigma + \kappa/(1+\beta)$ (as follows from the analysis in Appendix A.3). For any negative and odd integer h, simple algebra leads to $\hat{P}(\underline{\phi}, -1) = 0$ and $\partial\hat{P}/\partial z(\underline{\phi}, -1) = 1 + \beta + \kappa/[(1+\beta)\sigma] - [2(1+\beta) + \kappa/\sigma]h$. This last expression is generically non-zero. So, one root of the polynomial $\hat{P}(\underline{\phi}, z)$ is -1, and this root is of multiplicity one. The implicit-function theorem implies the existence of a continuously differentiable function $\phi \mapsto Z(\phi)$ such that one real root of $P(z)$ can be written as $Z(\phi)$ in the neighborhood of $\phi = \underline{\phi}$, with $Z(\underline{\phi}) = -1$ and

$$Z'(\underline{\phi}) = \frac{-\frac{\partial\hat{P}}{\partial\phi}(\underline{\phi}, -1)}{\frac{\partial\hat{P}}{\partial z}(\underline{\phi}, -1)} = \frac{\frac{1+\beta}{\sigma}}{1 + \beta + \frac{\kappa}{(1+\beta)\sigma} - \left[2(1+\beta) + \frac{\kappa}{\sigma}\right]h}.$$

For any negative and odd integer h, we have $Z'(\underline{\phi}) > 0$, and therefore the root of $P(z)$ goes from outside to inside $\mathcal{C}$ as ϕ crosses $\underline{\phi}$ from below. It is the only root that crosses $\mathcal{C}$ as ϕ goes through $\underline{\phi}$. Indeed, any root $z \in \mathbb{C}$ having this property must satisfy $\hat{P}(\underline{\phi}, z) = 0$, which implies $|Q(z)/(z-\beta)| = 2 + \kappa/[(1+\beta)\sigma]$ and hence $z = -1$ (since $\min_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})/(\tilde{z}-\beta)| = 2 + \kappa/[(1+\beta)\sigma]$ and $\arg\min_{\tilde{z}\in\mathcal{C}} |Q(\tilde{z})/(\tilde{z}-\beta)| = \{-1\}$, as follows from the analysis in Appendix A.3). So, the number of roots of $P(z)$ inside $\mathcal{C}$ increases by exactly one, from $p = 1 < \nu$ to $p = 2 = \nu$, and the determinacy status moves from M to D, as ϕ crosses ϕ_W from below. For any negative and odd integer h, thus, we get determinacy for ϕ just above $\underline{\phi}$. As a consequence, H_D is unbounded below.

A.6 Proof of Lemma 1

I start with the case of a policy-instrument peg ($\phi = 0$). In this case, the dynamic system boils down to $\mathbb{E}_t\{\mathbf{\Delta}(L^{-1})\mathbf{A}(L)\mathbf{X}_t\} = \mathbf{0}$. The characteristic polynomial of this system is the same as the characteristic polynomial of the corresponding perfect-foresight system. The latter system is $\mathbf{A}(L)\mathbf{X}_t = \mathbf{0}$. Since $\det[\mathbf{A}(0)] \neq 0$, using a standard result in time-series analysis (see, e.g., Hamilton, 1994, Proposition 10.1, Page 259), I get that $P(z)$, the reciprocal polynomial of this characteristic polynomial, is equal to $Q(z) := \det[\mathbf{A}(z)]$.

Since $\det[\mathbf{A}(0)] \neq 0$, the dynamic system can be rewritten as $\mathbb{E}_t\{\mathbf{\Delta}(L^{-1})\tilde{\mathbf{A}}(L)\tilde{\mathbf{X}}_t\} = \mathbf{0}$, where

$\tilde{\mathbf{A}}(z) := \mathbf{A}(z)[\mathbf{A}(0)]^{-1}$ and $\tilde{\mathbf{X}}_t := \mathbf{A}(0)\mathbf{X}_t$. Let $\tilde{X}_{j,t}$ denote the j^{th} element of $\tilde{\mathbf{X}}_t$ for $j \in \{1, ..., n\}$. The non-predetermined variables of the system are the variables $\mathbb{E}_t\{\tilde{X}_{j,t+k_j}\}$ for all $j \in \{1, ..., n\}$ such that $\delta_j \geq 1$ and all $k_j \in \{1, ..., \delta_j\}$. Their number, ν, is equal to $\delta := \sum_{j=1}^{n} \delta_j$.

I now turn to the case in which $\phi \neq 0$. In this case, the characteristic polynomial of the dynamic system is still the same as the characteristic polynomial of the corresponding perfect-foresight system, but the latter system is now

$$\begin{bmatrix} \mathbf{A}(L) & L^{-\gamma}\mathbf{B}(L) \\ -\phi L^{-h}\mathbf{V}(L) & 1 \end{bmatrix} \begin{bmatrix} \mathbf{X}_t \\ i_t \end{bmatrix} = \mathbf{0}.$$

Except possibly for a zero-measure set of values of ϕ, I can use the same standard result in time-series analysis as above. I get that there exists $k \in \mathbb{Z}$ such that $P(z)$, the reciprocal polynomial of the characteristic polynomial, is

$$P(z) = z^k \det \begin{bmatrix} \mathbf{A}(z) & z^{-\gamma}\mathbf{B}(z) \\ -\phi z^{-h}\mathbf{V}(z) & 1 \end{bmatrix}.$$

Using the Laplace expansion and the notations introduced in the main text, I rewrite $P(z)$ as $P(z) = z^k\{\det[\mathbf{A}(z)] - \phi z^{-\gamma-h}W(z)\} = z^k[Q(z) + \phi z^{m-h}R(z)]$. As a reciprocal polynomial, $P(z)$ is such that $P(0) \neq 0$; moreover, we have $Q(0) \neq 0$ and $R(0) \neq 0$; as a consequence, we get $k = \max(0, h-m)$, and thus $P(z) = Q(z)z^{\max(0,h-m)} + \phi R(z)z^{\max(0,m-h)}$.

The number of non-predetermined variables, ν, is equal to δ when h is lower than or equal to a certain threshold, and it increases one-for-one with h when h is higher than this threshold. This threshold is equal to the highest value of h for which $P(0)$ depends on $Q(0)$, i.e. for which the most forward variable in the dynamic system is the same as under a policy-instrument peg (except in the zero-measure case where $\phi = -Q(0)/R(0)$). This value is m, and thus $\nu = \delta + \max(0, h-m)$.

A.7 Proof of Proposition 5

The proof of Proposition 5 is essentially a generalization of the proof of Proposition 1, using this time $P(z) = Q(z)z^{\max(0,h-m)} + \phi R(z)z^{\max(0,m-h)}$ and $\nu = \delta + \max(0, h-m)$ (as stated in Lemma 1).

Point (a). I apply Rouché's theorem to $\mathcal{J} = \mathcal{C}$, $P_b(z) = Q(z)z^{\max(0,h-m)}$, and $P_s(z) = \phi R(z)z^{\max(0,m-h)}$. For any $|\phi| < \underline{\phi}$ and any $z \in \mathcal{C}$, we have

$$\left|Q(z)z^{\max(0,h-m)}\right| = |Q(z)| \geq \min_{\tilde{z}\in\mathcal{C}} \left|\frac{Q(\tilde{z})}{R(\tilde{z})}\right| |R(z)| = \underline{\phi}\,|R(z)| > |\phi R(z)| = \left|\phi R(z)z^{\max(0,m-h)}\right|.$$

So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}$ as $Q(z)z^{\max(0,h-m)}$, i.e. that $p = q + \max(0, h-m)$. Since $\nu = \delta + \max(0, h-m)$, we get, for any $|\phi| < \underline{\phi}$ and any $h \in \mathbb{Z}$: (i) if $q \leq \delta - 1$, then $p < \nu$ and $S(\phi, h) = M$; (ii) if $q = \delta$, then $p = \nu$ and $S(\phi, h) = D$; and (iii) if $q \geq \delta + 1$, then $p > \nu$ and $S(\phi, h) = E$.

Point (b). I apply Rouché's theorem to $\mathcal{J} = \mathcal{C}$, $P_b(z) = \phi R(z) z^{\max(0,m-h)}$, and $P_s(z) = Q(z) z^{\max(0,h-m)}$. For any $|\phi| > \bar{\phi}$ and any $z \in \mathcal{C}$, we have

$$\left|\phi R(z) z^{\max(0,m-h)}\right| = |\phi R(z)| > \bar{\phi}\,|R(z)| = \max_{\tilde{z}\in\mathcal{C}} \left|\frac{Q(\tilde{z})}{R(\tilde{z})}\right| |R(z)| \geq |Q(z)| = \left|Q(z) z^{\max(0,h-m)}\right|.$$

So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}$ as $\phi R(z) z^{\max(0,m-h)}$, i.e. that $p = r + \max(0, m-h)$. Since $\nu = \delta + \max(0, h-m)$, we get, for any $|\phi| > \bar{\phi}$: (i) if $h \leq h^* - 1$, then $p > \nu$ and $S(\phi, h) = E$; (ii) if $h = h^*$, then $p = \nu$ and $S(\phi, h) = D$; and (iii) if $h \geq h^* + 1$, then $p < \nu$ and $S(\phi, h) = M$.

Points (d)(i) and (d)(ii). For $h \leq m$, we have $\nu = \delta$ and $P(z) = Q(z) + \phi R(z) z^{m-h}$. I proceed in four steps.

In the first step, I show that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, all but $q + d_R - r$ roots of $P(z)$ converge uniformly to $\mathcal{C}$ as $h \to -\infty$. I get this result by applying Rouché's theorem twice. Since $q_{\mathcal{C}} = r_{\mathcal{C}} = 0$, I can consider an arbitrary $\epsilon \in (0,1)$ such that neither $Q(z)$ nor $R(z)$ has any root inside the annulus whose borders are $\mathcal{C}_{1-\epsilon}$ and $\mathcal{C}_{1+\epsilon}$ (where again, for any $r \in \mathbb{R}_+$, $\mathcal{C}_r$ denotes the circle of radius r centered at the origin of the complex plane).

I first apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1-\epsilon}$, $P_b(z) = Q(z)$, and $P_s(z) = \phi R(z) z^{m-h}$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \leq \underline{h}_{1-\epsilon} := m + \min\left\{0, \left\lfloor \frac{\log\left(\min_{\tilde{z}\in\mathcal{C}_{1-\epsilon}} |Q(\tilde{z})|\right) - \log\left(\bar{\phi}\max_{\tilde{z}\in\mathcal{C}_{1-\epsilon}} |R(\tilde{z})|\right)}{-\log(1-\epsilon)} \right\rfloor\right\},$$

and any $z \in \mathcal{C}_{1-\epsilon}$, we have

$$|Q(z)| \geq \min_{\tilde{z}\in\mathcal{C}_{1-\epsilon}} |Q(\tilde{z})| \geq \bar{\phi} \max_{\tilde{z}\in\mathcal{C}_{1-\epsilon}} |R(\tilde{z})| (1-\epsilon)^{m-h} \geq \bar{\phi}\left|R(z) z^{m-h}\right| > \left|\phi R(z) z^{m-h}\right|,$$

where the second inequality follows from the definition of $\underline{h}_{1-\epsilon}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}_{1-\epsilon}$ as $Q(z)$. Therefore, $P(z)$ has also exactly q roots inside $\mathcal{C}_{1-\epsilon}$ for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \leq \underline{h}_{1-\epsilon}$.

I then apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1+\epsilon}$, $P_b(z) = \phi R(z) z^{m-h}$, and $P_s(z) = Q(z)$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \leq \underline{h}_{1+\epsilon} := m + \min\left\{0, \left\lfloor \frac{\log\left(\underline{\phi}\min_{\tilde{z}\in\mathcal{C}_{1+\epsilon}} |R(\tilde{z})|\right) - \log\left(\max_{\tilde{z}\in\mathcal{C}_{1+\epsilon}} |Q(\tilde{z})|\right)}{\log(1+\epsilon)} \right\rfloor\right\},$$

and any $z \in \mathcal{C}_{1+\epsilon}$, we have

$$\left|\phi R(z) z^{m-h}\right| = |\phi R(z)| (1+\epsilon)^{m-h} > \underline{\phi} \min_{\tilde{z}\in\mathcal{C}_{1+\epsilon}} |R(\tilde{z})| (1+\epsilon)^{m-h} \geq \max_{\tilde{z}\in\mathcal{C}_{1+\epsilon}} |Q(\tilde{z})| \geq |Q(z)|,$$

where the last but one inequality follows from the definition of $\underline{h}_{1+\epsilon}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}_{1+\epsilon}$ as $\phi R(z) z^{m-h}$. Therefore, $P(z)$ has exactly $r + m - h$ roots inside $\mathcal{C}_{1+\epsilon}$ for any $h \leq \underline{h}_{1+\epsilon}$. As a consequence, for any $|\phi| \in (\underline{\phi}, \bar{\phi})$

and any $h \leq \min\left(\underline{h}_{1-\epsilon}, \underline{h}_{1+\epsilon}\right)$, $P(z)$ has exactly $r+m-h-q$ roots inside the annulus whose borders are $\mathcal{C}_{1-\epsilon}$ and $\mathcal{C}_{1+\epsilon}$. Now, the degree of $P(z)$ is d_R+m-h when $h \leq m+d_R-d_Q$. So, we eventually get that for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \leq \min\left(\underline{h}_{1-\epsilon}, \underline{h}_{1+\epsilon}, m+d_R-d_Q\right)$, all but $q+d_R-r$ roots of $P(z)$ lie between $\mathcal{C}_{1-\epsilon}$ and $\mathcal{C}_{1+\epsilon}$. We conclude that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, all but $q+d_R-r$ roots of $P(z)$ converge uniformly to $\mathcal{C}$ as $h \to -\infty$.

In the second step, I show that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, the roots of $P(z)$ uniformly converging to $\mathcal{C}$ as $h \to -\infty$ converge in distribution to the uniform distribution on $\mathcal{C}$. This result is a direct consequence of the Erdős-Turán theorem (stated in Appendix A.1). Applying this theorem to $\tilde{P}(z) = P(z)$, and using the result of the previous step, I thus get that all but $q+d_R-r$ roots of $P(z)$ uniformly converge in distribution to the uniform distribution on $\mathcal{C}$ as $h \to -\infty$, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$.

In the third step, I show that the number of roots of $P(z)$ inside $\mathcal{C}$ grows unboundedly as $h \to -\infty$, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$. Since $|\phi| > \underline{\phi}$, there exists an arc $\mathcal{A}$ of $\mathcal{C}$ such that $\forall z \in \mathcal{A}$, $|\phi R(z)| > |Q(z)|$. For any $r \in \mathbb{R}_+$, let $\mathcal{A}_r$ denote the image of $\mathcal{A}$ under the homothety whose center is the origin of the complex plane and whose ratio is r (so that in particular $\mathcal{A}_1 = \mathcal{A}$). By continuity, there exists $\varepsilon \in (0,1)$ such that $|\phi R(z)| > |Q(z)|$ for all z on the Jordan curve $\mathcal{J}_{1+\varepsilon}$ made of $\mathcal{A}$, $\mathcal{A}_{1+\varepsilon}$, and the two radial line segments joining the endpoints of $\mathcal{A}$ and $\mathcal{A}_{1+\varepsilon}$ (see Figure A.1 in Appendix A.1). I apply Rouché's theorem to $\mathcal{J} = \mathcal{J}_{1+\varepsilon}$, $P_b(z) = \phi R(z) z^{m-h}$, and $P_s(z) = Q(z)$. For any $h \leq m$ and any $z \in \mathcal{J}_{1+\varepsilon}$, we have

$$\left|\phi R(z) z^{m-h}\right| \geq |\phi R(z)| > |Q(z)|.$$

So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{J}_{1+\varepsilon}$ as $\phi R(z) z^{m-h}$. Therefore, $P(z)$ has at most d_R roots inside $\mathcal{J}_{1+\varepsilon}$ for any $h \leq m$. (Figure A.1 represents the case in which $P(z)$ has no root inside $\mathcal{J}_{1+\varepsilon}$; we necessarily get this case if ε is sufficiently small.) Using the results of the first two steps, we get that the number of roots of $P(z)$ inside the Jordan curve $\mathcal{J}_{1-\varepsilon}$ made of $\mathcal{A}_{1-\varepsilon}$, $\mathcal{A}$, and the two radial line segments joining the endpoints of $\mathcal{A}_{1-\varepsilon}$ and $\mathcal{A}$, grows unboundedly as $h \to -\infty$. As a result, p grows unboundedly as $h \to -\infty$. Thus, there exists $\underline{h}(|\phi|)$ such that $p > \delta = \nu$ and $S(\phi, h) = E$ for all $h \leq \underline{h}(|\phi|)$.

In the fourth step, I just note that for any $\varepsilon \in \left(0, \bar{\phi} - \underline{\phi}\right)$ and any $|\phi| \in (\underline{\phi} + \varepsilon, \bar{\phi})$, there exists, by continuity, $\ell(\varepsilon) > 0$ such that the arc $\mathcal{A}$ can be chosen of length higher than $\ell(\varepsilon)$. As a result, $\underline{h}(|\phi|)$ can be chosen a bounded function of $|\phi|$ for $|\phi| \in (\underline{\phi} + \varepsilon, \bar{\phi})$.

Points (c)(i) and (c)(ii). For $h \geq m$, we have $\nu = \delta + h - m$ and $P(z) = Q(z) z^{h-m} + \phi R(z)$. I follow the same four steps as in the proof of Points (d)(i) and (d)(ii) above, with some variants.

In the first step, I show that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, all but $r + d_Q - q$ roots of $P(z)$ converge uniformly to $\mathcal{C}$ as $h \to +\infty$. I get this result by applying Rouché's theorem twice. Since $q_{\mathcal{C}} = r_{\mathcal{C}} = 0$, I can consider an arbitrary $\epsilon \in (0,1)$ such that neither $Q(z)$ nor $R(z)$ has any root inside the annulus whose borders are $\mathcal{C}_{1-\epsilon}$ and $\mathcal{C}_{1+\epsilon}$.

I first apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1-\epsilon}$, $P_b(z) = \phi R(z)$, and $P_s(z) = Q(z)z^{h-m}$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \geq \bar{h}_{1-\epsilon} := m + \max\left\{0, \left\lceil \frac{\log\left(\max_{\tilde{z}\in\mathcal{C}_{1-\epsilon}} |Q(\tilde{z})|\right) - \log\left(\underline{\phi}\min_{\tilde{z}\in\mathcal{C}_{1-\epsilon}} |R(\tilde{z})|\right)}{-\log(1-\epsilon)} \right\rceil\right\},$$

and any $z \in \mathcal{C}_{1-\epsilon}$, we have

$$|\phi R(z)| > \underline{\phi} \min_{\tilde{z}\in\mathcal{C}_{1-\epsilon}} |R(\tilde{z})| \geq \max_{\tilde{z}\in\mathcal{C}_{1-\epsilon}} |Q(\tilde{z})| (1-\epsilon)^{h-m} \geq \left|Q(z)z^{h-m}\right|,$$

where the second inequality follows from the definition of $\bar{h}_{1-\epsilon}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}_{1-\epsilon}$ as $\phi R(z)$. Therefore, $P(z)$ has also exactly r roots inside $\mathcal{C}_{1-\epsilon}$ for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \geq \bar{h}_{1-\epsilon}$.

I then apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1+\epsilon}$, $P_b(z) = Q(z)z^{h-m}$, and $P_s(z) = \phi R(z)$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \geq \bar{h}_{1+\epsilon} := m + \max\left\{0, \left\lceil \frac{\log\left(\bar{\phi}\max_{\tilde{z}\in\mathcal{C}_{1+\epsilon}} |R(\tilde{z})|\right) - \log\left(\min_{\tilde{z}\in\mathcal{C}_{1+\epsilon}} |Q(\tilde{z})|\right)}{\log(1+\epsilon)} \right\rceil\right\},$$

and any $z \in \mathcal{C}_{1+\epsilon}$, we have

$$\left|Q(z)z^{h-m}\right| = |Q(z)| (1+\epsilon)^{h-m} \geq \min_{\tilde{z}\in\mathcal{C}_{1+\epsilon}} |Q(\tilde{z})| (1+\epsilon)^{h-m} \geq \bar{\phi} \max_{\tilde{z}\in\mathcal{C}_{1+\epsilon}} |R(\tilde{z})| > |\phi R(z)|,$$

where the last but one inequality follows from the definition of $\bar{h}_{1+\epsilon}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{C}_{1+\epsilon}$ as $Q(z)z^{h-m}$. Therefore, $P(z)$ has exactly $q+h-m$ roots inside $\mathcal{C}_{1+\epsilon}$ for any $h \geq \bar{h}_{1+\epsilon}$. As a consequence, for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \geq \max\left(\bar{h}_{1-\epsilon}, \bar{h}_{1+\epsilon}\right)$, $P(z)$ has exactly $q+h-m-r$ roots inside the annulus whose borders are $\mathcal{C}_{1-\epsilon}$ and $\mathcal{C}_{1+\epsilon}$. Now, the degree of $P(z)$ is $d_Q + h - m$ when $h \geq m + d_R - d_Q$. So, we eventually get that for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \geq \max\left(\bar{h}_{1-\epsilon}, \bar{h}_{1+\epsilon}, m + d_R - d_Q\right)$, all but $r + d_Q - q$ roots of $P(z)$ lie between $\mathcal{C}_{1-\epsilon}$ and $\mathcal{C}_{1+\epsilon}$. We conclude that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, all but $r + d_Q - q$ roots of $P(z)$ converge uniformly to $\mathcal{C}$ as $h \to +\infty$.

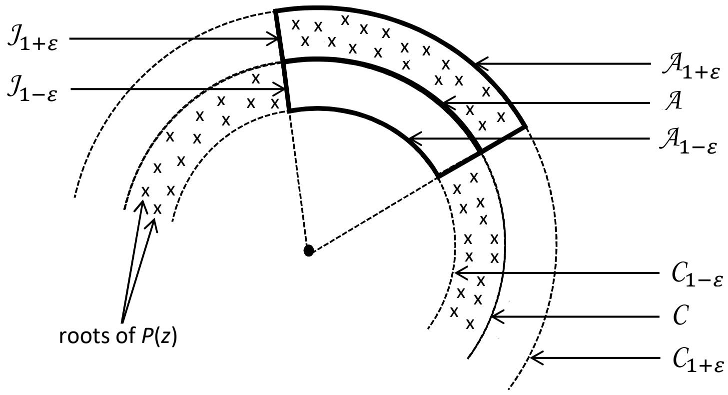
In the second step, I show that for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, the roots of $P(z)$ uniformly converging to $\mathcal{C}$ as $h \to +\infty$ converge in distribution to the uniform distribution on $\mathcal{C}$. This result is, again, a direct consequence of the Erdős-Turán theorem (stated in Appendix A.1). Applying this theorem to $\tilde{P}(z) = P(z)$, and using the result of the previous step, I thus get that all but $r + d_Q - q$ roots of $P(z)$ uniformly converge in distribution to the uniform distribution on $\mathcal{C}$ as $h \to +\infty$, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$.

In the third step, I show that the ratio p/ν is lower than 1 as $h \to +\infty$, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$. Since $|\phi| > \underline{\phi}$, there exists an arc $\mathcal{A}$ of $\mathcal{C}$ such that $\forall z \in \mathcal{A}$, $|\phi R(z)| > |Q(z)|$. By continuity, there exists $\varepsilon \in (0,1)$ such that $|\phi R(z)| > |Q(z)|$ for all z on the Jordan curve $\mathcal{J}_{1-\varepsilon}$ (defined above and represented in Figure A.3). I apply Rouché's theorem to $\mathcal{J} = \mathcal{J}_{1-\varepsilon}$, $P_b(z) = \phi R(z)$, and $P_s(z) = Q(z)z^{h-m}$. For any $h \geq m$ and any $z \in \mathcal{J}_{1-\varepsilon}$, we have

$$|\phi R(z)| > |Q(z)| \geq \left|Q(z)z^{h-m}\right|.$$

So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{J}_{1-\varepsilon}$ as $\phi R(z)$. Therefore, $P(z)$ has at most d_R roots inside $\mathcal{J}_{1-\varepsilon}$ for any $h \geq m$. (Figure A.3 represents the case in which $P(z)$ has no root inside $\mathcal{J}_{1-\varepsilon}$; we necessarily get this case if ε is sufficiently small.)

Figure A.3: Roots of $P(z)$ as $h \to +\infty$



Using the results of the first two steps, we get that the ratio of the number of roots of $P(z)$ inside the Jordan curve $\mathcal{J}_{1+\varepsilon}$ (defined above and represented in Figure A.3) to the total number of roots of $P(z)$ converges to $\ell(\mathcal{A})/(2\pi)$ as $h \to +\infty$, where again $\ell(.)$ denotes the standard length operator (i.e., the Lebesgue measure on $\mathcal{C}$). So, as $h \to +\infty$, the ratio of the number of roots of $P(z)$ outside $\mathcal{C}$ to the total number of roots of $P(z)$ is bounded away from 0; or, equivalently, the ratio of the number of roots of $P(z)$ inside $\mathcal{C}$ to the total number of roots of $P(z)$, i.e. the ratio p/d_P, is bounded away from 1. Since the ratio of the number of non-predetermined variables to the total number of roots of $P(z)$, i.e. the ratio ν/d_P, converges to 1 as $h \to +\infty$ (given that both ν and d_P increase one-for-one with h), we eventually get that the ratio p/ν is lower than 1 as $h \to +\infty$. Thus, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$, there exists $\bar{h}(|\phi|)$ such that $p < \nu$ and $S(\phi, h) = M$ for all $h \geq \bar{h}(|\phi|)$.

In the fourth step, I just note that for any $\varepsilon \in \left(0, \bar{\phi} - \underline{\phi}\right)$ and any $|\phi| \in (\underline{\phi} + \varepsilon, \bar{\phi})$, there exists, by continuity, $\ell(\varepsilon) > 0$ such that the arc $\mathcal{A}$ can be chosen of length higher than $\ell(\varepsilon)$. As a result, $\bar{h}(|\phi|)$ can be chosen a bounded function of $|\phi|$ for $|\phi| \in (\underline{\phi} + \varepsilon, \bar{\phi})$.

Point (c)(iii). For $h \geq m + \max(0, d_R - d_Q)$, we have $\nu = \delta + h - m$, $P(z) = Q(z)z^{h-m} + \phi R(z)$, and $d_P = d_Q + h - m$. Consider a Jordan curve $\mathcal{J}_o$ (where the subscript "o" stands for "outside $\mathcal{C}$") that: (i) lies entirely outside $\mathcal{C}$, (ii) surrounds the $d_Q - q$ roots of $Q(z)$ outside $\mathcal{C}$ (if $d_Q - q \geq 1$), and (iii) does not surround $\mathcal{C}$. I apply Rouché's theorem to $\mathcal{J} = \mathcal{J}_o$, $P_b(z) = Q(z)z^{h-m}$, and $P_s(z) = \phi R(z)$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \geq \bar{h} := m + \max\left\{0, d_R - d_Q, \left\lceil \frac{\log\left(\bar{\phi} \max_{\tilde{z} \in \mathcal{J}_o} |R(\tilde{z})|\right) - \log\left(\min_{\tilde{z} \in \mathcal{J}_o} |Q(\tilde{z})|\right)}{\log\left(\min_{\tilde{z} \in \mathcal{J}_o} |\tilde{z}|\right)} \right\rceil \right\},$$

and any $z \in \mathcal{J}_o$, we have

$$\left|Q(z)z^{h-m}\right| \geq \min_{\tilde{z} \in \mathcal{J}_o} \left|Q(\tilde{z})\,\tilde{z}^{h-m}\right| \geq \left(\min_{\tilde{z} \in \mathcal{J}_o} |Q(\tilde{z})|\right) \left(\min_{\tilde{z} \in \mathcal{J}_o} |\tilde{z}|\right)^{h-m} \geq \bar{\phi} \max_{\tilde{z} \in \mathcal{J}_o} |R(\tilde{z})| > |\phi R(z)|,$$

where the last but one inequality follows from the definition of $\bar{h}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{J}_o$ as $Q(z)z^{h-m}$. Therefore, $P(z)$ has exactly $d_Q - q$ roots inside $\mathcal{J}_o$, and hence at least $d_Q - q$ roots outside $\mathcal{C}$. We thus get $p \leq d_P - (d_Q - q) = h - m + q = (q - \delta) + \nu$. Therefore, if $q \leq \delta - 1$, then $p < \nu$ and consequently $S(\phi, h) = M$ for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \geq \bar{h}$.

Point (d)(iii). For $h \leq m$, we have $\nu = \delta$ and $P(z) = Q(z) + \phi R(z) z^{m-h}$. Consider a Jordan curve $\mathcal{J}_i$ (where the subscript "i" stands for "inside $\mathcal{C}$") that: (i) lies entirely inside $\mathcal{C}$, and (ii) surrounds the q roots of $Q(z)$ inside $\mathcal{C}$ (if $q \geq 1$). I apply Rouché's theorem to $\mathcal{J} = \mathcal{J}_i$, $P_b(z) = Q(z)$, and $P_s(z) = \phi R(z) z^{m-h}$. For any $|\phi| \in (\underline{\phi}, \bar{\phi})$, any

$$h \leq \underline{h} := m + \min\left\{0, \left\lfloor \frac{\log\left(\min_{\tilde{z}\in\mathcal{J}_i} |Q(\tilde{z})|\right) - \log\left(\bar{\phi} \max_{\tilde{z}\in\mathcal{J}_i} |R(\tilde{z})|\right)}{-\log\left(\max_{\tilde{z}\in\mathcal{J}_i} |\tilde{z}|\right)} \right\rfloor\right\},$$

and any $z \in \mathcal{J}_i$, we have

$$|Q(z)| \geq \min_{\tilde{z}\in\mathcal{J}_i} |Q(\tilde{z})| \geq \bar{\phi} \max_{\tilde{z}\in\mathcal{J}_i} |R(\tilde{z})| \left(\max_{\tilde{z}\in\mathcal{J}_i} |\tilde{z}|\right)^{m-h} \geq \bar{\phi} \max_{\tilde{z}\in\mathcal{J}_i} \left|R(\tilde{z})\, \tilde{z}^{m-h}\right| > \left|\phi R(z) z^{m-h}\right|,$$

where the second inequality follows from the definition of $\underline{h}$. So, Rouché's theorem implies that $P(z)$ has the same number of roots inside $\mathcal{J}_i$ as $Q(z)$. Therefore, $P(z)$ has exactly q roots inside $\mathcal{J}_i$, and hence at least q roots inside $\mathcal{C}$. We thus get $p \geq q = (q - \delta) + \nu$. Therefore, if $q \geq \delta + 1$, then $p > \nu$ and consequently $S(\phi, h) = E$ for any $|\phi| \in (\underline{\phi}, \bar{\phi})$ and any $h \leq \underline{h}$.

A.8 Proof of Proposition 6

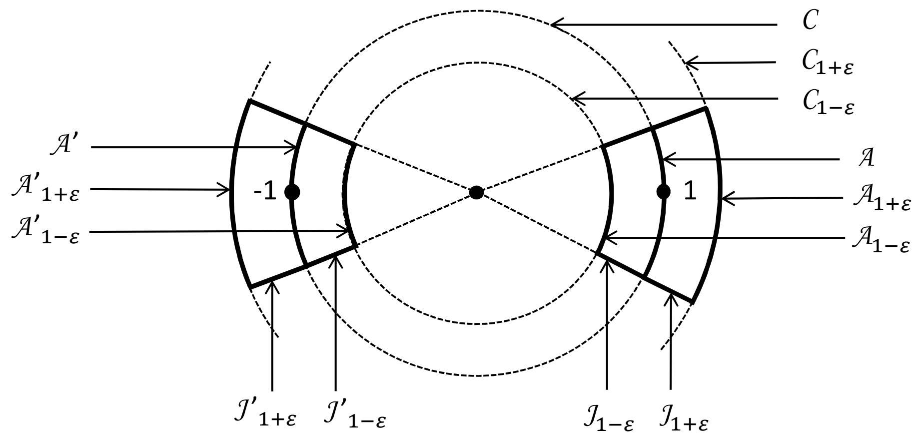
Points (a) and (b)(i). These points straightforwardly follow from Points (a), (c)(iii), and (d)(iii) of Proposition 5.

Point (b)(ii). The proof of this point is similar to (part of) the proof of Point (b) of Proposition 4. If $A_{min} \subset \mathcal{C} \setminus \{-1, 1\}$, then $|Q(z)| > \underline{\phi} |R(z)|$ for $z \in \{-1, 1\}$. So, by continuity, there exist $\epsilon \in (0, \bar{\phi} - \underline{\phi})$ and two open arcs $\mathcal{A}$ and $\mathcal{A}'$ of $\mathcal{C}$ such that: (i) $1 \in \mathcal{A}$, (ii) $-1 \in \mathcal{A}'$, and (iii) $\forall |\phi| \in (\underline{\phi}, \underline{\phi} + \epsilon)$, $\forall z \in \mathcal{A} \cup \mathcal{A}'$, $|Q(z)| > |\phi R(z)|$.

For any $r \in \mathbb{R}_+$, let $\mathcal{A}_r$ (resp. $\mathcal{A}'_r$) denote the image of $\mathcal{A}$ (resp. $\mathcal{A}'$) under the homothety whose center is the origin of the complex plane and whose ratio is r, so that in particular $\mathcal{A}_1 = \mathcal{A}$ (resp. $\mathcal{A}'_1 = \mathcal{A}'$). In addition, for any $r \in \mathbb{R}_+ \setminus \{0\}$, let $\mathcal{J}_r$ (resp. $\mathcal{J}'_r$) denote the Jordan curve made of $\mathcal{A}$ (resp. $\mathcal{A}'$), $\mathcal{A}_r$ (resp. $\mathcal{A}'_r$), and the two radial line segments joining the endpoints of $\mathcal{A}$ (resp. $\mathcal{A}'$) and $\mathcal{A}_r$ (resp. $\mathcal{A}'_r$).

By continuity, there exists $\varepsilon \in (0, 1)$ such that: (i) neither $Q(z)$ nor $R(z)$ has any root inside the annulus whose borders are $\mathcal{C}_{1-\varepsilon}$ and $\mathcal{C}_{1+\varepsilon}$, (ii) $\forall |\phi| \in (\underline{\phi}, \underline{\phi} + \epsilon)$, $\forall h \leq m$, $\forall z \in \mathcal{J}_{1-\varepsilon} \cup \mathcal{J}'_{1-\varepsilon}$, $|Q(z)| > \left|\phi R(z) z^{m-h}\right|$, and (iii) $\forall |\phi| \in (\underline{\phi}, \underline{\phi} + \epsilon)$, $\forall h \geq m$, $\forall z \in \mathcal{J}_{1+\varepsilon} \cup \mathcal{J}'_{1+\varepsilon}$, $\left|Q(z) z^{h-m}\right| > |\phi R(z)|$ (see Figure A.4).

Figure A.4: Jordan curves $\mathcal{J}_{1-\varepsilon}$, $\mathcal{J}_{1+\varepsilon}$, $\mathcal{J}'_{1-\varepsilon}$, and $\mathcal{J}'_{1+\varepsilon}$



I first apply Rouché's theorem to $P_b(z) = Q(z)$, $P_s(z) = \phi R(z) z^{m-h}$, and (alternatively) $\mathcal{J}_{1-\varepsilon}$ and $\mathcal{J}'_{1-\varepsilon}$. I obtain that $\forall |\phi| \in (\underline{\phi}, \underline{\phi} + \epsilon)$, $\forall h \leq m$, $P(z)$ has no root inside $\mathcal{J}_{1-\varepsilon}$ and no root inside $\mathcal{J}'_{1-\varepsilon}$. Therefore, $P(z)$ has no real root between $\mathcal{C}_{1-\varepsilon}$ and $\mathcal{C}$. Now, we know from Appendix A.7 that there exists $\underline{h}_{1-\varepsilon} \leq m$ such that $\forall |\phi| \in (\underline{\phi}, \bar{\phi})$, $\forall h \leq \underline{h}_{1-\varepsilon}$, $P(z)$ has exactly q roots inside $\mathcal{C}_{1-\varepsilon}$. As a result, $\forall |\phi| \in (\underline{\phi}, \underline{\phi} + \epsilon)$, $\forall h \leq \underline{h}_{1-\varepsilon}$, $P(z)$ has exactly $p - q$ roots between $\mathcal{C}_{1-\varepsilon}$ and $\mathcal{C}$, and none of them is real, so $p - q$ is even. Therefore, if $q - \delta$ is odd, then $p - \delta$ is odd too. Since $\nu = \delta$ for $h \leq \underline{h}_{1-\varepsilon}$, $p - \nu$ is odd as well. As a consequence, $p \neq \nu$, and hence $S(\phi, h) \neq D$, for all $|\phi| \in (\underline{\phi}, \underline{\phi} + \epsilon)$ and all $h \leq \underline{h}_{1-\varepsilon}$. Together with Point (d)(ii) of Proposition 5, this result implies that H_D is bounded below.

I then apply Rouché's theorem to $P_b(z) = Q(z) z^{h-m}$, $P_s(z) = \phi R(z)$, and (alternatively) $\mathcal{J}_{1+\varepsilon}$ and $\mathcal{J}'_{1+\varepsilon}$. I obtain that $\forall |\phi| \in (\underline{\phi}, \underline{\phi} + \epsilon)$, $\forall h \geq m$, $P(z)$ has no root inside $\mathcal{J}_{1+\varepsilon}$ and no root inside $\mathcal{J}'_{1+\varepsilon}$. Therefore, $P(z)$ has no real root between $\mathcal{C}$ and $\mathcal{C}_{1+\varepsilon}$. Now, we know from Appendix A.7 that there exists $\bar{h}_{1+\varepsilon} \geq m$ such that $\forall |\phi| \in (\underline{\phi}, \bar{\phi})$, $\forall h \geq \bar{h}_{1+\varepsilon}$, $P(z)$ has exactly $q + h - m$ roots inside $\mathcal{C}_{1+\varepsilon}$. As a result, $\forall |\phi| \in (\underline{\phi}, \underline{\phi} + \epsilon)$, $\forall h \geq \bar{h}_{1+\varepsilon}$, $P(z)$ has exactly $q + h - m - p$ roots between $\mathcal{C}$ and $\mathcal{C}_{1+\varepsilon}$, and none of them is real, so $q + h - m - p$ is even. Therefore, if $q - \delta$ is odd, then $\delta + h - m - p$ is odd too. Since $\nu = \delta + h - m$ for $h \geq \bar{h}_{1+\varepsilon}$, $\nu - p$ is odd as well. As a consequence, $p \neq \nu$, and hence $S(\phi, h) \neq D$, for all $|\phi| \in (\underline{\phi}, \underline{\phi} + \epsilon)$ and all $h \geq \bar{h}_{1+\varepsilon}$. Together with Point (c)(ii) of Proposition 5, this result implies that H_D is bounded above.

Point (b)(iii). I consider four alternative cases in turn. First, if $A_{min} = \{1\}$ and $Q(1)R(1) < 0$, then Point (e) of Proposition 7 (proved in Appendix A.9 below without using Proposition 6) implies that H_D is unbounded below (resp. above) if $q = \delta - 1$ (resp. $q = \delta + 1$). Second, if $A_{min} = \{1\}$ and $Q(1)R(1) > 0$, then the same reasoning as in Appendix A.9 below, this time for $\phi_W = -\underline{\phi}$ instead of $\phi_W = \underline{\phi}$, straightforwardly implies that H_D is, again, unbounded below (resp. above) if $q = \delta - 1$ (resp. $q = \delta + 1$).

The third case that I consider is the case in which $A_{min} = \{-1\}$ and $Q(-1)R(-1) > 0$, and hence $\underline{\phi} = Q(-1)/R(-1)$. In this case, I rewrite $P(z)$ as a function of two variables:

$\hat{P}(\phi, z) := Q(z)z^{\max(0,h-m)} + \phi R(z)z^{\max(0,m-h)}$, where $(\phi, z) \in \mathbb{R} \times \mathbb{C}$. For any $h \in \mathbb{Z}$ such that $h - m$ is odd, simple algebra leads to $\hat{P}(\underline{\phi}, -1) = 0$ and

$$\frac{\partial \hat{P}}{\partial z}(\underline{\phi}, -1) = (-1)^{\max(0,m-h)}\underline{\phi}R(-1)\left(h - \widetilde{h}\right),$$

where $\widetilde{h} := m+Q'(-1)/Q(-1)-R'(-1)/R(-1)$. This expression for $\partial\hat{P}/\partial z(\underline{\phi}, -1)$ is generically non-zero (it can be zero only if $Q'(1)/Q(1) - R'(1)/R(1)$ is an integer, and I ignore this zero-measure case). So, one root of the polynomial $\hat{P}(\underline{\phi}, z)$ is -1, and this root is of multiplicity one. The implicit-function theorem implies the existence of a continuously differentiable function $\phi \mapsto Z(\phi)$ such that one real root of $P(z)$ can be written as $Z(\phi)$ in the neighborhood of $\phi = \underline{\phi}$, with $Z(\underline{\phi}) = -1$ and

$$Z'(\underline{\phi}) = \frac{-\frac{\partial \hat{P}}{\partial \phi}(\underline{\phi}, -1)}{\frac{\partial \hat{P}}{\partial z}(\underline{\phi}, -1)} = \frac{-1}{\underline{\phi}\left(h - \widetilde{h}\right)}.$$

This root of $P(z)$ crosses $\mathcal{C}$ at point -1 as ϕ goes through $\underline{\phi}$. It is the only root that crosses $\mathcal{C}$ as ϕ goes through $\underline{\phi}$. Indeed, any root $z \in \mathbb{C}$ having this property must satisfy $\hat{P}(\underline{\phi}, z) = 0$, which implies $|Q(z)| = \underline{\phi}|R(z)|$ and hence $z = -1$ (since $A_{min} = \{-1\}$).

If $h < \widetilde{h}$, then $Z'(\underline{\phi}) > 0$, and therefore the root of $P(z)$ goes from outside to inside $\mathcal{C}$ as ϕ crosses $\underline{\phi}$ from below. So, the number of roots of $P(z)$ inside $\mathcal{C}$, p, increases by exactly one as ϕ crosses $\underline{\phi}$ from below. We know from Appendix A.7 that this number is $p = q + \max(0, h - m)$ for ϕ just below $\underline{\phi}$. We also know from Lemma 1 that $\nu = \delta + \max(0, h - m)$. So, if $q = \delta - 1$, then we move from $p < \nu$ to $p = \nu$, and hence from $S(\phi, h) = M$ to $S(\phi, h) = D$, as ϕ crosses $\underline{\phi}$ from below, for any $h \in \mathbb{Z}$ such that $h - m$ is odd and such that $h < \widetilde{h}$. Therefore, H_D is unbounded below.

Alternatively, if $h > \widetilde{h}$, then $Z'(\underline{\phi}) < 0$, and therefore the root of $P(z)$ goes this time from inside to outside $\mathcal{C}$ as ϕ crosses $\underline{\phi}$ from below. So, p decreases by exactly one as ϕ crosses $\underline{\phi}$ from below. Again, we know from Appendix A.7 that $p = q + \max(0, h - m)$ for ϕ just below $\underline{\phi}$, and we know from Lemma 1 that $\nu = \delta + \max(0, h - m)$. So, if $q = \delta + 1$, then we move from $p > \nu$ to $p = \nu$, and hence from $S(\phi, h) = E$ to $S(\phi, h) = D$, as ϕ crosses $\underline{\phi}$ from below, for any $h \in \mathbb{Z}$ such that $h - m$ is odd and such that $h > \widetilde{h}$. Therefore, H_D is unbounded above.

The fourth and last case that I consider is the case in which $A_{min} = \{-1\}$ and $Q(-1)R(-1) < 0$, and hence $\underline{\phi} = -Q(-1)/R(-1)$. In this case, the analysis and the conclusion are exactly the same as in the third case, except that "$h - m$ is odd" should be replaced by "$h - m$ is even."

A.9 Proof of Proposition 7

Points (a)-(d) and (e)(i). These points straightforwardly follow from the definitions of H_D and ϕ_W, Points (a)-(b) of Proposition 5, and the restriction $\phi_W > 0$.

Point (e)(ii). The proof of this point is essentially a generalization of the proofs of Point

(c) of Proposition 3 and Point (f)(ii) of Proposition 4. I assume that $A_{min} = \{1\}$ (as stated in this point). I rewrite $P(z)$ as a function of two variables: $\hat{P}(\phi, z) := Q(z)z^{\max(0,h-m)} + \phi R(z)z^{\max(0,m-h)}$, where $(\phi, z) \in \mathbb{R} \times \mathbb{C}$. Simple algebra leads to $\hat{P}(\phi_W, 1) = 0$ and

$$\frac{\partial \hat{P}}{\partial z}(\phi_W, 1) = Q(1)\,(h - h^{**})\,.$$

This last expression is generically non-zero (it can be zero only if $Q'(1)/Q(1) - R'(1)/R(1)$ is an integer, and I ignore this zero-measure case). So, one root of the polynomial $\hat{P}(\phi_W, z)$ is 1, and this root is of multiplicity one. The implicit-function theorem implies the existence of a continuously differentiable function $\phi \mapsto Z(\phi)$ such that one real root of $P(z)$ can be written as $Z(\phi)$ in the neighborhood of $\phi = \phi_W$, with $Z(\phi_W) = 1$ and

$$Z'(\phi_W) = \frac{-\frac{\partial \hat{P}}{\partial \phi}(\phi_W, 1)}{\frac{\partial \hat{P}}{\partial z}(\phi_W, 1)} = \frac{1}{\phi_W\,(h - h^{**})}.$$

This root of $P(z)$ crosses $\mathcal{C}$ at point 1 as ϕ goes through ϕ_W. It is the only root that crosses $\mathcal{C}$ as ϕ goes through ϕ_W. Indeed, any root $z \in \mathbb{C}$ having this property must satisfy $\hat{P}(\phi_W, z) = 0$, which implies $|Q(z)| = \underline{\phi}\,|R(z)|$ and hence $z = 1$ (since $A_{min} = \{1\}$).

For any $h < h^{**}$, we have $Z'(\phi_W) < 0$, and therefore the root of $P(z)$ goes from outside to inside $\mathcal{C}$ as ϕ crosses ϕ_W from below. So, the number of roots of $P(z)$ inside $\mathcal{C}$, p, increases by exactly one as ϕ crosses ϕ_W from below. We know from Appendix A.7 that this number is $p = q + \max(0, h - m)$ for ϕ just below $\underline{\phi} = \phi_W$. We also know from Lemma 1 that $\nu = \delta + \max(0, h - m)$. So, if $q = \delta - 1$, then we move from $p < \nu$ to $p = \nu$, and hence from $S(\phi, h) = M$ to $S(\phi, h) = D$, as ϕ crosses ϕ_W from below; in this case, the Taylor principle is locally necessary and sufficient for determinacy. Alternatively, if $q \leq \delta - 2$ (resp. $q \geq \delta$), then we get $p < \nu$ and $S(\phi, h) = M$ (resp. $p > \nu$ and $S(\phi, h) = E$) for ϕ just above ϕ_W, and the Taylor principle is not locally necessary and sufficient for determinacy.

For any $h > h^{**}$, we have $Z'(\phi_W) > 0$, and therefore the root of $P(z)$ goes this time from inside to outside $\mathcal{C}$ as ϕ crosses ϕ_W from below. So, p decreases by exactly one as ϕ crosses ϕ_W from below. Again, we know from Appendix A.7 that $p = q + \max(0, h - m)$ for ϕ just below $\underline{\phi} = \phi_W$, and we know from Lemma 1 that $\nu = \delta + \max(0, h - m)$. So, if $q = \delta + 1$, then we move from $p > \nu$ to $p = \nu$, and hence from $S(\phi, h) = E$ to $S(\phi, h) = D$, as ϕ crosses ϕ_W from below; in this case, the Taylor principle is locally necessary and sufficient for determinacy. Alternatively, if $q \leq \delta$ (resp. $q \geq \delta + 2$), then we get $p < \nu$ and $S(\phi, h) = M$ (resp. $p > \nu$ and $S(\phi, h) = E$) for ϕ just above ϕ_W, and the Taylor principle is not locally necessary and sufficient for determinacy.

A.10 Proof of Proposition 8

Point (b1). I consider an arbitrary $\varepsilon \in (0, 1)$ such that the only roots of $R(z)$ between $\mathcal{C}_{1-\varepsilon}$ (included) and $\mathcal{C}_{1+\varepsilon}$ (included) are the $r_{\mathcal{C}}$ roots of $R(z)$ on $\mathcal{C}$. I proceed in four steps.

In the first step, I show that $\forall h \leq \underline{h}^* - 1$, $\exists \phi_h > 0$, $\forall |\phi| > \phi_h$, $S(\phi, h) = E$. I apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1-\varepsilon}$, $P_b(z) = \phi R(z) z^{\max(0,m-h)}$, and $P_s(z) = Q(z) z^{\max(0,h-m)}$. For any $h \in \mathbb{Z}$, any

$$|\phi| > \max_{\tilde{z} \in \mathcal{C}_{1-\varepsilon}} \left| \frac{Q(\tilde{z})}{R(\tilde{z})} \right| (1-\varepsilon)^{h-m}, \tag{A.2}$$

and any $z \in \mathcal{C}_{1-\varepsilon}$, we have $\left|\phi R(z) z^{\max(0,m-h)}\right| > \left|Q(z) z^{\max(0,h-m)}\right|$. So, $P(z)$ has the same number of roots inside $\mathcal{C}_{1-\varepsilon}$ as $\phi R(z) z^{\max(0,m-h)}$. Therefore, $P(z)$ has exactly $r + \max(0, m-h) = \nu + (\underline{h}^* - h)$ roots inside $\mathcal{C}_{1-\varepsilon}$, and hence at least $\nu + (\underline{h}^* - h)$ roots inside $\mathcal{C}$. For $h \leq \underline{h}^* - 1$ and $|\phi| > \phi_h := \max_{\tilde{z} \in \mathcal{C}_{1-\varepsilon}} |Q(\tilde{z}) / R(\tilde{z})| (1-\varepsilon)^{h-m}$, thus, we have $p > \nu$ and $S(\phi, h) = E$.

In the second step, I show that $\forall h \geq \bar{h}^* + 1$, $\exists \phi_h > 0$, $\forall |\phi| > \phi_h$, $S(\phi, h) = M$. I apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1+\varepsilon}$, $P_b(z) = \phi R(z) z^{\max(0,m-h)}$, and $P_s(z) = Q(z) z^{\max(0,h-m)}$. For any $h \in \mathbb{Z}$, any

$$|\phi| > \max_{\tilde{z} \in \mathcal{C}_{1+\varepsilon}} \left| \frac{Q(\tilde{z})}{R(\tilde{z})} \right| (1+\varepsilon)^{h-m}, \tag{A.3}$$

and any $z \in \mathcal{C}_{1+\varepsilon}$, we have $\left|\phi R(z) z^{\max(0,m-h)}\right| > \left|Q(z) z^{\max(0,h-m)}\right|$. So, $P(z)$ has the same number of roots inside $\mathcal{C}_{1+\varepsilon}$ as $\phi R(z) z^{\max(0,m-h)}$. Therefore, $P(z)$ has exactly $r + r_{\mathcal{C}} + \max(0, m-h) = \nu + \left(\bar{h}^* - h\right)$ roots inside $\mathcal{C}_{1+\varepsilon}$, and hence at most $\nu + \left(\bar{h}^* - h\right)$ roots inside $\mathcal{C}$. For $h \geq \bar{h}^* + 1$ and $|\phi| > \phi_h := \max_{\tilde{z} \in \mathcal{C}_{1+\varepsilon}} |Q(\tilde{z}) / R(\tilde{z})| (1+\varepsilon)^{h-m}$, thus, we have $p < \nu$ and $S(\phi, h) = M$.

In the third step, I show that $\forall h \in \{\underline{h}^*, ..., \bar{h}^*\}$, $\exists \phi_h > 0$, $\forall |\phi| > \phi_h$, $S(\phi, h)$ may depend on ϕ only through the sign of ϕ. It is easy to see, with Rouché's theorem, that for any given $h \in \{\underline{h}^*, ..., \bar{h}^*\}$, as $|\phi| \to +\infty$, d_R roots of $P(z)$ converge to the roots of $R(z)$, while the other roots of $P(z)$ (if any) either converge to 0, or diverge to $+\infty$ in modulus. So, for any given $h \in \{\underline{h}^*, ..., \bar{h}^*\}$, there exists $\phi_h > 0$ such that the number of roots of $P(z)$ inside $\mathcal{C}$ and the number of roots of $P(z)$ outside $\mathcal{C}$ are constant functions of ϕ for $\phi \in (\phi_h, +\infty)$, and constant functions of ϕ for $\phi \in (-\infty, -\phi_h)$. These numbers p and $d_P - p$ depend on how many of the $r_{\mathcal{C}}$ roots of $P(z)$ converging to the $r_{\mathcal{C}}$ roots of $R(z)$ on $\mathcal{C}$ converge from inside $\mathcal{C}$, and how many of them converge from outside $\mathcal{C}$.

In the fourth step, I use the following lemmas:

Lemma 2: *$\exists \phi_{-\infty} > 0$, $\exists \underline{h} \leq \underline{h}^* - 1$, $\forall |\phi| > \phi_{-\infty}$, $\forall h \leq \underline{h}$, $S(\phi, h) = E$.*

Lemma 3: *$\exists \phi_{+\infty} > 0$, $\exists \bar{h} \geq \bar{h}^* + 1$, $\forall |\phi| > \phi_{+\infty}$, $\forall h \geq \bar{h}$, $S(\phi, h) = M$.*

Proofs: See Appendix A.11 for Lemma 2, and Appendix A.12 for Lemma 3. ■

These two lemmas, together with the results of the previous steps, straightforwardly imply Point (b1) with $\bar{\phi} := \max\left\{\phi_{-\infty}, \max\left\{\phi_h | \underline{h} \leq h \leq \bar{h}\right\}, \phi_{+\infty}\right\}$.

Point (b2). I assume that $r_{\mathcal{C}} = 1$ and $R(1) = 0$ (as stated in this point), which implies in particular that $\bar{h}^* = \underline{h}^* + 1$. We know from the proof of Point (b1) above that, for $|\phi|$ sufficiently large, $P(z)$ has exactly $\nu + (\underline{h}^* - h)$ roots inside $\mathcal{C}_{1-\varepsilon}$ and exactly $\nu + \left(\bar{h}^* - h\right)$ roots inside $\mathcal{C}_{1+\varepsilon}$.

Therefore, for $h = \underline{h}^*$ (resp. $h = \bar{h}^*$) and $|\phi|$ sufficiently large, $P(z)$ has exactly ν (resp. $\nu - 1$) roots inside $\mathcal{C}_{1-\varepsilon}$ and exactly $\nu + 1$ (resp. ν) roots inside $\mathcal{C}_{1+\varepsilon}$.

For $\phi \neq 0$, $P(z)$ has the same roots as $\epsilon Q(z) z^{\max(0,h-m)} + R(z) z^{\max(0,m-h)}$, where $\epsilon := 1/\phi$. I rewrite the latter polynomial as a function of two variables: $\hat{P}(\epsilon, z)$. I extend the domain of this function to allow ϵ to be zero: $(\epsilon, z) \in \mathbb{R} \times \mathbb{C}$. We have $\hat{P}(0,1) = 0$ and $\partial \hat{P}/\partial z(0,1) = R'(1) \neq 0$, where the inequality follows from $r_{\mathcal{C}} = 1$ (which implies that 1 is not a multiple root of $R(z)$). So, one root of the polynomial $\hat{P}(0,z)$ is 1, and this root is of multiplicity one. The implicit-function theorem implies the existence of a continuously differentiable function $\epsilon \mapsto Z(\epsilon)$ such that one real root of $\hat{P}(\epsilon, z)$ can be written as $Z(\epsilon)$ in the neighborhood of $\epsilon = 0$, with $Z(0) = 1$ and

$$Z'(0) = \frac{-\frac{\partial \hat{P}}{\partial \epsilon}(0,1)}{\frac{\partial \hat{P}}{\partial z}(0,1)} = \frac{-Q(1)}{R'(1)}.$$

This real root of $\hat{P}(\epsilon, z)$ crosses $\mathcal{C}$ at point 1 as ϵ goes through 0.

For $-Q(1)/R'(1) < 0$, or equivalently $Q(1)R'(1) > 0$, we have $Z'(0) < 0$; so, for $|\epsilon|$ sufficiently small, this real root of $\hat{P}(\epsilon, z)$ is just above 1 if $\epsilon < 0$, and just below 1 if $\epsilon > 0$. Therefore, for $\phi = 1/\epsilon$ sufficiently large in absolute value, one real root of $P(z)$ is just above 1 if $\phi < 0$, and just below 1 if $\phi > 0$. As a result, for $h = \underline{h}^*$ (resp. $h = \bar{h}^*$) and $|\phi|$ sufficiently large, $P(z)$ has exactly ν (resp. $\nu - 1$) roots inside $\mathcal{C}$ if $\phi < 0$, and exactly $\nu + 1$ (resp. ν) roots inside $\mathcal{C}$ if $\phi > 0$. Thus, $(S(\phi, \underline{h}^*), S(\phi, \bar{h}^*)) = (D, M)$ if $\phi < 0$, and $(S(\phi, \underline{h}^*), S(\phi, \bar{h}^*)) = (E, D)$ if $\phi > 0$. This result holds for $|\phi|$ sufficiently large and in particular, by construction of $\bar{\phi}$, for $|\phi| > \bar{\phi}$.

Alternatively, for $Q(1)R'(1) < 0$, we have $Z'(0) > 0$; so, for $|\epsilon|$ sufficiently small, this real root of $\hat{P}(\epsilon, z)$ is just below 1 if $\epsilon < 0$, and just above 1 if $\epsilon > 0$. Therefore, for $\phi = 1/\epsilon$ sufficiently large in absolute value, one real root of $P(z)$ is just below 1 if $\phi < 0$, and just above 1 if $\phi > 0$. As a result, for $h = \underline{h}^*$ (resp. $h = \bar{h}^*$) and $|\phi|$ sufficiently large, $P(z)$ has exactly $\nu + 1$ (resp. ν) roots inside $\mathcal{C}$ if $\phi < 0$, and exactly ν (resp. $\nu - 1$) roots inside $\mathcal{C}$ if $\phi > 0$. Thus, $(S(\phi, \underline{h}^*), S(\phi, \bar{h}^*)) = (E, D)$ if $\phi < 0$, and $(S(\phi, \underline{h}^*), S(\phi, \bar{h}^*)) = (D, M)$ if $\phi > 0$. This result holds for $|\phi|$ sufficiently large and in particular, by construction of $\bar{\phi}$, for $|\phi| > \bar{\phi}$.

Other points. Proposition 8 states that Points (a) and (c)-(d) of Proposition 5 still hold. The proofs of Points (a), (c)(iii), and (d)(iii) are exactly the same as in Appendix A.7 (with $\bar{\phi}$ in these proofs now being defined as in the proof of Point (b1) above). Points (d)(i) and (d)(ii) are a direct consequence of the following two results: first, as I show in Appendix A.11, $\forall b > 0$, $\exists \underline{h}_{1+\epsilon} \in \mathbb{Z}$, $\forall |\phi| > \underline{\phi} + b$, $\forall h \leq \min\left(\underline{h}_{1+\epsilon}, m - d_Q - 1\right)$, if $x \geq \bar{x}$ then $S(\phi, h) = E$, where x denotes the number of roots of $P(z)$ whose angular coordinate belongs to a certain non-degenerate interval (which depends neither on ϕ nor on h), while $\bar{x}$ is a given positive integer (which depends neither on ϕ nor on h); and second, as straightforwardly implied by the Erdős-Turán theorem, the angular coordinates of the roots of $P(z)$ converge in distribution to the uniform distribution on $(0, 2\pi)$ as $h \to -\infty$, for any given $|\phi| \in (\underline{\phi}, \bar{\phi})$. Similarly, Points (c)(i) and (c)(ii) are a direct consequence of the following two results: first, as I show in Appendix

A.12, $\forall b>0$, $\exists \bar{h}_{1-\epsilon}\in\mathbb{Z}$, $\forall|\phi|>\underline{\phi}+b$, $\forall h\geq\max\left(\bar{h}_{1-\epsilon},m+d_R+1\right)$, if $x\geq\bar{x}$ then $S(\phi,h)=M$, where x denotes again the number of roots of $P(z)$ whose angular coordinate belongs to a certain non-degenerate interval (which depends neither on ϕ nor on h), while $\bar{x}$ is again a given positive integer (which depends neither on ϕ nor on h); and second, as straightforwardly implied by the Erdős-Turán theorem, the angular coordinates of the roots of $P(z)$ converge in distribution to the uniform distribution on $(0,2\pi)$ as $h\to+\infty$, for any given $|\phi|\in(\underline{\phi},\bar{\phi})$.

Proposition 8 also states that Points (a)-(b) of Proposition 6 still hold. The proofs of these points are exactly the same as in Appendix A.8. The proof of Point (b)(ii) of Proposition 6, in particular, uses two results established in Appendix A.7: these results still hold, and their proof is unchanged. Similarly, the proof of Point (b)(iii) of Proposition 6 is unchanged because $R(1)\neq 0$ when $A_{min}=\{1\}$, and $R(-1)\neq 0$ when $A_{min}=\{-1\}$. Finally, Proposition 8 states that if $R(1)\neq 0$ and $\phi_W>0$, then Points (a)-(b) and (e) of Proposition 7 still hold. The proofs of these points are exactly the same as in Appendix A.9.

A.11 Proof of Lemma 2

Let b be an arbitrary positive real number (with "b" standing for "buffer"). There exists an arc $\mathcal{A}$ of $\mathcal{C}$ such that $\forall|\phi|>\underline{\phi}+b$, $\forall z\in\mathcal{A}$, $|\phi R(z)|>|Q(z)|$. By continuity, there also exists $\epsilon>0$ such that: (i) $\forall|\phi|>\underline{\phi}+b$, $\forall z\in\mathcal{J}_{1+\epsilon}$, $|\phi R(z)|>|Q(z)|$ (where again $\mathcal{J}_{1+\epsilon}$ denotes the Jordan curve made of $\mathcal{A}$, $\mathcal{A}_{1+\epsilon}$, and the two radial line segments joining the endpoints of $\mathcal{A}$ and $\mathcal{A}_{1+\epsilon}$), and (ii) $R(z)$ has no root between $\mathcal{C}$ (excluded) and $\mathcal{C}_{1+\epsilon}$ (included). I apply Rouché's theorem to $\mathcal{J}=\mathcal{J}_{1+\epsilon}$, $P_b(z)=\phi R(z)z^{m-h}$, and $P_s(z)=Q(z)$. For any $|\phi|>\underline{\phi}+b$, any $h\leq m$, and any $z\in\mathcal{J}_{1+\epsilon}$, we have

$$\left|\phi R(z)z^{m-h}\right|\geq|\phi R(z)|>|Q(z)|.$$

So, $P(z)$ has the same number of roots inside $\mathcal{J}_{1+\epsilon}$ as $\phi R(z)z^{m-h}$; therefore, $P(z)$ has no root inside $\mathcal{J}_{1+\epsilon}$. I also apply Rouché's theorem to $\mathcal{J}=\mathcal{C}_{1+\epsilon}$, $P_b(z)=\phi R(z)z^{m-h}$, and $P_s(z)=Q(z)$. For any $|\phi|>\underline{\phi}+b$, any

$$h\leq\underline{h}_{1+\epsilon}:=m+\min\left\{0,\left\lfloor\frac{\log\left(\underline{\phi}\min_{\tilde{z}\in\mathcal{C}_{1+\epsilon}}|R(\tilde{z})|\right)-\log\left(\max_{\tilde{z}\in\mathcal{C}_{1+\epsilon}}|Q(\tilde{z})|\right)}{\log(1+\epsilon)}\right\rfloor\right\},$$

and any $z\in\mathcal{C}_{1+\epsilon}$, we have

$$\left|\phi R(z)z^{m-h}\right|=|\phi R(z)|(1+\epsilon)^{m-h}>\underline{\phi}\min_{\tilde{z}\in\mathcal{C}_{1+\epsilon}}|R(\tilde{z})|(1+\epsilon)^{m-h}\geq\max_{\tilde{z}\in\mathcal{C}_{1+\epsilon}}|Q(\tilde{z})|\geq|Q(z)|,$$

where the last but one inequality follows from the definition of $\underline{h}_{1+\epsilon}$. So, $P(z)$ has the same number of roots inside $\mathcal{C}_{1+\epsilon}$ as $\phi R(z)z^{m-h}$; therefore, $P(z)$ has exactly $r+r_{\mathcal{C}}+m-h$ roots inside $\mathcal{C}_{1+\epsilon}$. In the following, I restrict my attention to $|\phi|>\underline{\phi}+b$ and $h\leq\min\left(\underline{h}_{1+\epsilon},m-d_Q-1\right)$. This restriction implies that $P(z)$ has no root inside $\mathcal{J}_{1+\epsilon}$ and has exactly $r+r_{\mathcal{C}}+m-h$ roots inside $\mathcal{C}_{1+\epsilon}$. Moreover, $h\leq m-d_Q-1$ implies $d_P=d_R+m-h$, so $P(z)$ has exactly $d_P-(r+r_{\mathcal{C}}+m-h)=d_R-r-r_{\mathcal{C}}$ roots outside $\mathcal{C}_{1+\epsilon}$.

Let $\underline{\alpha}$ and $\bar{\alpha}$ denote the angular coordinates of the endpoints of $\mathcal{A}$, with $0 < \underline{\alpha} < \bar{\alpha} < 2\pi$ (I am assuming for simplicity that $1 \notin \mathcal{A}$; this assumption is made without any loss in generality, since $\mathcal{A}$ can always be chosen such that $1 \notin \mathcal{A}$). Let x denote the number of roots of $P(z)$ whose angular coordinate belongs to $(\underline{\alpha}, \bar{\alpha})$. These x roots can have a modulus lower than 1 or higher than $1+\epsilon$, but not between 1 and $1+\epsilon$, since $P(z)$ has no root inside $\mathcal{J}_{1+\epsilon}$. Moreover, the number of roots of modulus higher than $1+\epsilon$ among these x roots cannot exceed $d_R - r - r_{\mathcal{C}}$, which is the total number of roots of $P(z)$ outside $\mathcal{C}_{1+\epsilon}$. Therefore, the number of roots of modulus lower than 1 among these x roots is at least $x - (d_R - r - r_{\mathcal{C}})$. Thus, if $x \geq \bar{x} := d_R - r - r_{\mathcal{C}} + \delta + 1$, then the number of roots of modulus lower than 1 among these x roots is at least $\delta + 1$, so $p \geq \delta + 1 > \delta = \nu$ and $S(\phi, h) = E$.

I now establish a condition on $|\phi|$ and h that implies $x \geq \bar{x}$ and hence $S(\phi, h) = E$. Applying the Erdős-Turán theorem (stated in Appendix A.1) to $\tilde{P}(z) = P(z)$, I get $x - \bar{x} \geq f(|\phi|, h)$ with

$$f(|\phi|, h) := \left(\frac{\bar{\alpha} - \underline{\alpha}}{2\pi}\right)(d_R + m - h) - \bar{x} - 16\sqrt{(d_R + m - h)\log\left(\frac{\sum_{k=0}^{d_Q}|q_k| + |\phi|\sum_{k=0}^{d_R}|r_k|}{\sqrt{|q_0 r_{d_R}\phi|}}\right)},$$

where q_j for $0 \leq j \leq d_Q$ denotes the coefficient of z^j in $Q(z)$, and r_j for $0 \leq j \leq d_R$ denotes the coefficient of z^j in $R(z)$. For $|\phi| \geq 1$ and $h \leq h_1 := \lfloor d_R + m - 4\pi\bar{x}/(\bar{\alpha} - \underline{\alpha})\rfloor$, moreover, we have $f(|\phi|, h) \geq 16 g(|\phi|, h)$ with

$$g(|\phi|, h) := \sqrt{K_1}\,(d_R + m - h) - \sqrt{(d_R + m - h)\log\left(K_2\sqrt{|\phi|}\right)},$$

where $K_1 := [(\bar{\alpha} - \underline{\alpha})/(64\pi)]^2$ and $K_2 := (\sum_{k=0}^{d_Q}|q_k| + \sum_{k=0}^{d_R}|r_k|)/\sqrt{|q_0 r_{d_R}|}$. So, for $|\phi| \geq 1$ and $h \leq h_1$, a sufficient condition for $f(|\phi|, h) \geq 0$, and hence $x \geq \bar{x}$, and hence $S(\phi, h) = E$, is $g(|\phi|, h) \geq 0$, or equivalently

$$\log|\phi| \leq (-2K_1)\,h + [2K_1(d_R + m) - 2\log K_2]. \tag{A.4}$$

For $h \leq \underline{h}^* - 1$, another sufficient condition for $S(\phi, h) = E$ is (A.2), which can be rewritten as

$$\log|\phi| > [\log(1 - \varepsilon)]\,h + [\log K_3 - m\log(1 - \varepsilon)], \tag{A.5}$$

where $K_3 := \max_{\tilde{z} \in \mathcal{C}_{1-\varepsilon}} |Q(\tilde{z})/R(\tilde{z})|$. The parameter $\varepsilon \in (0, 1)$ in (A.2) can be chosen arbitrarily small. I choose it lower than $1 - e^{-2K_1}$. Then, the two sufficient conditions (A.4) and (A.5) together imply that $S(\phi, h) = E$ for $|\phi| \geq 1$ and $h \leq \min(h_1, \underline{h}^* - 1, h_2)$, where

$$h_2 := \left\lfloor \frac{2K_1(d_R + m) + m\log(1 - \varepsilon) - 2\log K_2 - \log K_3}{2K_1 + \log(1 - \varepsilon)} \right\rfloor.$$

To conclude, taking into account the restrictions imposed earlier on $|\phi|$ and h, I thus get $S(\phi, h) = E$ for all $|\phi| > \phi_{-\infty}$ and all $h \leq \underline{h}$, where $\phi_{-\infty} := \max\left(\underline{\phi} + b, 1\right)$ and $\underline{h} := \min\left(\underline{h}_{1+\epsilon}, m - d_Q - 1, h_1, \underline{h}^* - 1, h_2\right)$.

A.12 Proof of Lemma 3

The proof of Lemma 3 is similar to that of Lemma 2. Let b be again an arbitrary positive real number (with "b" standing for "buffer"). There exists, again, an arc $\mathcal{A}$ of $\mathcal{C}$ such that $\forall\, |\phi| > \underline{\phi} + b$, $\forall z \in \mathcal{A}$, $|\phi R(z)| > |Q(z)|$. By continuity, there also exists $\epsilon > 0$ such that: (i) $\forall\, |\phi| > \underline{\phi} + b$, $\forall z \in \mathcal{J}_{1-\epsilon}$, $|\phi R(z)| > |Q(z)|$ (where again $\mathcal{J}_{1-\epsilon}$ denotes the Jordan curve made of $\mathcal{A}$, $\mathcal{A}_{1-\epsilon}$, and the two radial line segments joining the endpoints of $\mathcal{A}$ and $\mathcal{A}_{1-\epsilon}$), and (ii) $R(z)$ has no root between $\mathcal{C}_{1-\epsilon}$ (included) and $\mathcal{C}$ (excluded). I apply Rouché's theorem to $\mathcal{J} = \mathcal{J}_{1-\epsilon}$, $P_b(z) = \phi R(z)$, and $P_s(z) = Q(z) z^{h-m}$. For any $|\phi| > \underline{\phi} + b$, any $h \geq m$, and any $z \in \mathcal{J}_{1-\epsilon}$, we have

$$|\phi R(z)| > |Q(z)| \geq \left| Q(z) z^{h-m} \right|.$$

So, $P(z)$ has the same number of roots inside $\mathcal{J}_{1-\epsilon}$ as $\phi R(z)$; therefore, $P(z)$ has no root inside $\mathcal{J}_{1-\epsilon}$. I also apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1-\epsilon}$, $P_b(z) = \phi R(z)$, and $P_s(z) = Q(z) z^{h-m}$. For any $|\phi| > \underline{\phi} + b$, any

$$h \geq \bar{h}_{1-\epsilon} := m + \max\left\{0, \left\lceil \frac{\log\left(\max_{\tilde{z} \in \mathcal{C}_{1-\epsilon}} |Q(\tilde{z})|\right) - \log\left(\underline{\phi} \min_{\tilde{z} \in \mathcal{C}_{1-\epsilon}} |R(\tilde{z})|\right)}{-\log(1-\epsilon)} \right\rceil \right\},$$

and any $z \in \mathcal{C}_{1-\epsilon}$, we have

$$|\phi R(z)| > \underline{\phi} \min_{\tilde{z} \in \mathcal{C}_{1-\epsilon}} |R(\tilde{z})| \geq \max_{\tilde{z} \in \mathcal{C}_{1-\epsilon}} |Q(\tilde{z})| (1-\epsilon)^{h-m} \geq |Q(z)| (1-\epsilon)^{h-m} = \left| Q(z) z^{h-m} \right|,$$

where the second inequality follows from the definition of $\bar{h}_{1-\epsilon}$. So, $P(z)$ has the same number of roots inside $\mathcal{C}_{1-\epsilon}$ as $\phi R(z)$; therefore, $P(z)$ has exactly r roots inside $\mathcal{C}_{1-\epsilon}$. In the following, I restrict my attention to $|\phi| > \underline{\phi} + b$ and $h \geq \max\left(\bar{h}_{1-\epsilon}, m + d_R + 1\right)$. This restriction implies that $P(z)$ has no root inside $\mathcal{J}_{1-\epsilon}$ and has exactly r roots inside $\mathcal{C}_{1-\epsilon}$. Moreover, $h \geq m + d_R + 1$ implies $d_P = d_Q + h - m$.

Let $\underline{\alpha}$ and $\bar{\alpha}$ denote, again, the angular coordinates of the endpoints of $\mathcal{A}$, with $0 < \underline{\alpha} < \bar{\alpha} < 2\pi$ (I am again assuming, for simplicity and without any loss in generality, that $1 \notin \mathcal{A}$). Let x denote, again, the number of roots of $P(z)$ whose angular coordinate belongs to $(\underline{\alpha}, \bar{\alpha})$. These x roots can have a modulus lower than $1-\epsilon$ or higher than 1, but not between $1-\epsilon$ and 1, since $P(z)$ has no root inside $\mathcal{J}_{1-\epsilon}$. Moreover, the number of roots of modulus lower than $1-\epsilon$ among these x roots cannot exceed r, which is the total number of roots of $P(z)$ inside $\mathcal{C}_{1-\epsilon}$. Therefore, the number of roots of modulus higher than 1 among these x roots is at least $x - r$. Thus, if $x \geq \bar{x} := d_Q + r + 1 - \delta$, then the number of roots of modulus higher than 1 among these x roots is at least $d_Q + 1 - \delta$; so, the number of roots of $P(z)$ outside $\mathcal{C}$ is also at least $d_Q + 1 - \delta$; so, $p \leq d_P - (d_Q + 1 - \delta) = \delta + h - m - 1 = \nu - 1 < \nu$ and $S(\phi, h) = M$.

I now establish a condition on $|\phi|$ and h that implies $x \geq \bar{x}$ and hence $S(\phi, h) = M$. Applying the Erdős-Turán theorem (stated in Appendix A.1) to $\tilde{P}(z) = P(z)$, I get $x - \bar{x} \geq f(|\phi|, h)$ with

$$f(|\phi|, h) := \left(\frac{\bar{\alpha} - \underline{\alpha}}{2\pi}\right)(d_Q + h - m) - \bar{x} - 16\sqrt{(d_Q + h - m)\log\left(\frac{\sum_{k=0}^{d_Q}|q_k| + |\phi|\sum_{k=0}^{d_R}|r_k|}{\sqrt{|q_{d_Q} r_0 \phi|}}\right)},$$

where again q_j for $0 \leq j \leq d_Q$ denotes the coefficient of z^j in $Q(z)$, and r_j for $0 \leq j \leq d_R$ denotes the coefficient of z^j in $R(z)$. For $|\phi| \geq 1$ and $h \geq h_1 := \lceil m - d_Q + 4\pi\bar{x}/(\bar{\alpha} - \underline{\alpha})\rceil$, moreover, we have $f(|\phi|, h) \geq 16 g(|\phi|, h)$ with

$$g(|\phi|, h) := \sqrt{K_1}(d_Q + h - m) - \sqrt{(d_Q + h - m)\log\left(K_2\sqrt{|\phi|}\right)},$$

where $K_1 := [(\bar{\alpha} - \underline{\alpha})/(64\pi)]^2$ and $K_2 := (\sum_{k=0}^{d_Q}|q_k| + \sum_{k=0}^{d_R}|r_k|)/\sqrt{|q_{d_Q} r_0|}$. So, for $|\phi| \geq 1$ and $h \geq h_1$, a sufficient condition for $f(|\phi|, h) \geq 0$, and hence $x \geq \bar{x}$, and hence $S(\phi, h) = M$, is $g(|\phi|, h) \geq 0$, or equivalently

$$\log|\phi| \leq (2K_1)\, h + [2K_1(d_Q - m) - 2\log K_2]. \tag{A.6}$$

For $h \geq \bar{h}^* + 1$, another sufficient condition for $S(\phi, h) = M$ is (A.3), which can be rewritten as

$$\log|\phi| > [\log(1+\varepsilon)]\, h + [\log K_3 - m\log(1+\varepsilon)], \tag{A.7}$$

where $K_3 := \max_{\tilde{z} \in \mathcal{C}_{1+\varepsilon}} |Q(\tilde{z})/R(\tilde{z})|$. The parameter $\varepsilon \in (0, 1)$ in (A.3) can be chosen arbitrarily small. I choose it lower than $e^{2K_1} - 1$. Then, the two sufficient conditions (A.6) and (A.7) together imply that $S(\phi, h) = M$ for $|\phi| \geq 1$ and $h \geq \max(h_1, \bar{h}^* + 1, h_2)$, where

$$h_2 := \left\lceil \frac{2\log K_2 + \log K_3 - m\log(1+\varepsilon) - 2K_1(d_Q - m)}{2K_1 - \log(1+\varepsilon)} \right\rceil.$$

To conclude, taking into account the restrictions imposed earlier on $|\phi|$ and h, I thus get $S(\phi, h) = M$ for all $|\phi| > \phi_{+\infty}$ and all $h \geq \bar{h}$, where $\phi_{+\infty} := \max\left(\underline{\phi} + b, 1\right)$ and $\bar{h} := \max\left(\bar{h}_{1-\epsilon}, m + d_R + 1, h_1, \bar{h}^* + 1, h_2\right)$. Note finally that $h \geq m + d_R + 1$ and $d_R \geq r + r_{\mathcal{C}}$ imply $h \geq \bar{h}^* + 1$; so, $\bar{h}$ can be more simply written as $\bar{h} = \max\left(\bar{h}_{1-\epsilon}, m + d_R + 1, h_1, h_2\right)$.

A.13 Proof of Proposition 9

Points (e1)-(e4). I assume that $q_{\mathcal{C}} = 1$ and $Q(1) = 0$ (as stated in the proposition). I consider an arbitrary $\varepsilon \in (0, 1)$ such that the only root of $Q(z)$ between $\mathcal{C}_{1-\varepsilon}$ (included) and $\mathcal{C}_{1+\varepsilon}$ (included) is 1 (with multiplicity one). I first apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1-\varepsilon}$, $P_b(z) = Q(z)z^{\max(0,h-m)}$, and $P_s(z) = \phi R(z) z^{\max(0,m-h)}$. For any $h \in \mathbb{Z}$, any

$$|\phi| < \min_{\tilde{z} \in \mathcal{C}_{1-\varepsilon}} \left|\frac{Q(\tilde{z})}{R(\tilde{z})}\right| (1-\varepsilon)^{h-m},$$

and any $z \in \mathcal{C}_{1-\varepsilon}$, we have $\left|Q(z)z^{\max(0,h-m)}\right| > \left|\phi R(z) z^{\max(0,m-h)}\right|$. So, $P(z)$ has the same number of roots inside $\mathcal{C}_{1-\varepsilon}$ as $Q(z)z^{\max(0,h-m)}$. Therefore, $P(z)$ has exactly $q + \max(0, h-m) =$

$\nu + (q - \delta)$ roots inside $\mathcal{C}_{1-\varepsilon}$, and hence at least $\nu + (q - \delta)$ roots inside $\mathcal{C}$: $p \geq \nu + (q - \delta)$. Thus, if $q \geq \delta + 1$, then $p > \nu$ and $(S^+, S^-) = (E, E)$.

I then apply Rouché's theorem to $\mathcal{J} = \mathcal{C}_{1+\varepsilon}$, $P_b(z) = Q(z) z^{\max(0,h-m)}$, and $P_s(z) = \phi R(z) z^{\max(0,m-h)}$. For any $h \in \mathbb{Z}$, any

$$|\phi| < \min_{\tilde{z} \in \mathcal{C}_{1+\varepsilon}} \left| \frac{Q(\tilde{z})}{R(\tilde{z})} \right| (1 + \varepsilon)^{h-m},$$

and any $z \in \mathcal{C}_{1+\varepsilon}$, we have $\left| Q(z) z^{\max(0,h-m)} \right| > \left| \phi R(z) z^{\max(0,m-h)} \right|$. So, $P(z)$ has the same number of roots inside $\mathcal{C}_{1+\varepsilon}$ as $Q(z) z^{\max(0,h-m)}$. Therefore, $P(z)$ has exactly $q + 1 + \max(0, h - m) = \nu + (q + 1 - \delta)$ roots inside $\mathcal{C}_{1+\varepsilon}$, and hence at most $\nu + (q + 1 - \delta)$ roots inside $\mathcal{C}$: $p \leq \nu + (q + 1 - \delta)$. Thus, if $q \leq \delta - 2$, then $p < \nu$ and $(S^+, S^-) = (M, M)$.

For $q \in \{\delta - 1, \delta\}$, $S(\phi, h)$ depends on whether the unique root of $P(z)$ between $\mathcal{C}_{1-\varepsilon}$ and $\mathcal{C}_{1+\varepsilon}$ lies inside or outside $\mathcal{C}$. To answer this question, I rewrite $P(z)$ as a function of two variables: $\hat{P}(\phi, z) := Q(z) z^{\max(0,h-m)} + \phi R(z) z^{\max(0,m-h)}$, where $(\phi, z) \in \mathbb{R} \times \mathbb{C}$. We have $\hat{P}(0, 1) = 0$ and $\partial \hat{P} / \partial z(0, 1) = Q'(1) \neq 0$, where the inequality follows from $q_{\mathcal{C}} = 1$ (which implies that 1 is not a multiple root of $Q(z)$). So, one root of the polynomial $\hat{P}(0, z)$ is 1, and this root is of multiplicity one. The implicit-function theorem implies the existence of a continuously differentiable function $\phi \mapsto Z(\phi)$ such that one real root of $P(z)$ can be written as $Z(\phi)$ in the neighborhood of $\phi = 0$, with $Z(0) = 1$ and

$$Z'(0) = \frac{-\frac{\partial \hat{P}}{\partial \phi}(0, 1)}{\frac{\partial \hat{P}}{\partial z}(0, 1)} = \frac{-R(1)}{Q'(1)}.$$

This real root of $P(z)$ crosses $\mathcal{C}$ at point 1 as ϕ goes through 0.

For $-R(1)/Q'(1) < 0$, or equivalently $Q'(1)R(1) > 0$, we have $Z'(0) < 0$. So, for $|\phi|$ sufficiently small, this real root of $P(z)$ is just above 1 if $\phi < 0$, and just below 1 if $\phi > 0$. Therefore, $p = \nu + (q - \delta)$ if $\phi < 0$, and $p = \nu + (q + 1 - \delta)$ if $\phi > 0$. When $q = \delta - 1$, thus, we have $p < \nu$ if $\phi < 0$, and $p = \nu$ if $\phi > 0$; so, $(S^+, S^-) = (D, M)$. When $q = \delta$, we have $p = \nu$ if $\phi < 0$, and $p > \nu$ if $\phi > 0$; so, $(S^+, S^-) = (E, D)$.

Alternatively, for $Q'(1)R(1) < 0$, we have $Z'(0) > 0$. So, for $|\phi|$ sufficiently small, this real root of $P(z)$ is just below 1 if $\phi < 0$, and just above 1 if $\phi > 0$. Therefore, $p = \nu + (q + 1 - \delta)$ if $\phi < 0$, and $p = \nu + (q - \delta)$ if $\phi > 0$. When $q = \delta - 1$, thus, we have $p = \nu$ if $\phi < 0$, and $p < \nu$ if $\phi > 0$; so, $(S^+, S^-) = (M, D)$. When $q = \delta$, we have $p > \nu$ if $\phi < 0$, and $p = \nu$ if $\phi > 0$; so, $(S^+, S^-) = (D, E)$.

Other points. Points (e5)-(e6) of Proposition 9 are a straightforward consequence of Points (e1)-(e4) of this proposition. Proposition 9 states that Point (b) of Proposition 5 still holds. The proof of this point is exactly the same as in Appendix A.7. This point straightforwardly implies that Points (c)-(d) of Proposition 7 still hold as well.

Proposition 9 also states that Points (d)(i) and (d)(ii) of Proposition 5 still hold. The proof of these points is exactly the same as in Appendix A.7, except that: (i) the arbitrary $\epsilon \in (0, 1)$

in the first step of the proof should now be chosen such that the only roots of $Q(z)$ and $R(z)$ between $\mathcal{C}_{1-\varepsilon}$ (included) and $\mathcal{C}_{1+\varepsilon}$ (included) are the $q_{\mathcal{C}}$ roots of $Q(z)$ on $\mathcal{C}$; (ii) $|\phi| \in (\underline{\phi}, \bar{\phi})$ in this proof should be replaced by $|\phi| \in (\underline{\phi} + b, \bar{\phi})$, where b is an arbitrary positive real number (with "b" standing for "buffer"); and (iii) $\underline{\phi}$ should be replaced by $\underline{\phi} + b$ in the definition of $\underline{h}_{1+\epsilon}$ and in the inequality just below this definition.

Finally, Proposition 9 states that Points (c)(i) and (c)(ii) of Proposition 5 still hold. The proof of these points is exactly the same as in Appendix A.7, except that: (i) again, the arbitrary $\epsilon \in (0, 1)$ in the first step of the proof should now be chosen such that the only roots of $Q(z)$ and $R(z)$ between $\mathcal{C}_{1-\varepsilon}$ (included) and $\mathcal{C}_{1+\varepsilon}$ (included) are the $q_{\mathcal{C}}$ roots of $Q(z)$ on $\mathcal{C}$; (ii) again, $|\phi| \in (\underline{\phi}, \bar{\phi})$ in this proof should be replaced by $|\phi| \in (\underline{\phi} + b, \bar{\phi})$, where b is an arbitrary positive real number (with "b" standing for "buffer"); (iii) $\underline{\phi}$ should be replaced by $\underline{\phi} + b$ in the definition of $\bar{h}_{1-\epsilon}$ and in the inequality just below this definition; and (iv) in the first two steps of the proof, q should be replaced by $q + q_{\mathcal{C}}$.

A.14 Proof of Proposition 10

Proposition 10 states that Points (b1)-(b2) of Proposition 8 still hold. The proof of Point (b1) is exactly the same as in Appendix A.10; this proof rests on Lemmas 2 and 3; the proofs of these lemmas are exactly the same as in Appendices A.11 and A.12, except that $\underline{\phi}$ should be replaced by $\underline{\phi} + b$ in the definition of $\underline{h}_{1+\epsilon}$ and in the inequality just below this definition (in Appendix A.11), as well as in the definition of $\bar{h}_{1-\epsilon}$ and in the inequality just below this definition (in Appendix A.12). The proof of Point (b2) is also exactly the same as in Appendix A.10; in particular, we still have $Q(1) \neq 0$ in this proof because $Q(z)$ and $R(z)$ have no common root on $\mathcal{C}$.

Proposition 10 also states that Points (c)(i), (c)(ii), (d)(i), and (d)(ii) of Proposition 5 still hold. The proof of these points is exactly the same as in Appendix A.10; this proof uses two results established in Appendices A.11 and A.12; the proof of these results is unchanged except that, again, $\underline{\phi}$ should be replaced by $\underline{\phi} + b$ in the definition of $\underline{h}_{1+\epsilon}$ and in the inequality just below this definition (in Appendix A.11), as well as in the definition of $\bar{h}_{1-\epsilon}$ and in the inequality just below this definition (in Appendix A.12).

Finally, Proposition 10 states that if $q_{\mathcal{C}} = 1$ and $Q(1) = 0$, then Points (e1)-(e6) of Proposition 9 still hold. The proof of these points is exactly the same as in Appendix A.13; in particular, we still have $R(1) \neq 0$ in this proof because $Q(z)$ and $R(z)$ have no common root on $\mathcal{C}$.

A.15 Proof of Proposition 11

Under Rule (5) with $\phi \neq 0$, we have $P(z) = z^{\max(0,h-m)}[Q(z) + \phi R(z) z^{m-h}]$, as follows from Lemma 1. For any $j \in \{1, ..., J\}$, under the rule $i_t = \phi_j \mathbb{E}_t\{v_{j,t+h_j}\}$ with $\phi_j \neq 0$, we similarly have $P(z) = z^{\max(0,h_j-m_j)}[Q(z) + \phi_j R_j(z) z^{m_j-h_j}]$.

Under Rule (7) with $\phi = 0$, therefore, there exists $k_1 \in \mathbb{Z}$ such that $P(z) = z^{k_1}[Q(z) + \sum_{j=1}^{J} \phi_j R_j(z) z^{m_j-h_j}]$. As a reciprocal polynomial, $P(z)$ is such that $P(0) \neq 0$; moreover, we have $Q(0) \neq 0$ and $\forall j \in \{1, ..., J\}$, $R_j(0) \neq 0$; as a consequence, we get $k_1 = g := \max[0, \max_{j\in\{1,...,J\}}(h_j - m_j)]$, and thus $P(z) = \tilde{Q}(z) := z^g[Q(z) + \sum_{j=1}^{J} \phi_j R_j(z) z^{m_j-h_j}]$. In addition, the same argument as the one used at the end of Appendix A.6 implies that the number of non-predetermined variables, ν, is equal to $\tilde{\delta} := \delta + g$.

Under Rule (7) with $\phi \neq 0$, similarly, there exists $k_2 \in \mathbb{Z}$ such that $P(z) = z^{k_2}[Q(z) + \sum_{j=1}^{J} \phi_j R_j(z) z^{m_j-h_j} + \phi R(z) z^{m-h}]$. Again, as a reciprocal polynomial, $P(z)$ is such that $P(0) \neq 0$; moreover, we have $Q(0) \neq 0$, $\forall j \in \{1, ..., J\}$, $R_j(0) \neq 0$, and $R(0) \neq 0$; as a consequence, we get $k_2 = \max[0, \max_{j\in\{1,...,J\}}(h_j - m_j), h - m] = \max(g, h - m)$, and thus $P(z) = \tilde{Q}(z) z^{\max(0,h-\tilde{m})} + \phi R(z) z^{\max(0,\tilde{m}-h)}$, where $\tilde{m} := m + g$. In addition, the same argument as the one used at the end of Appendix A.6 implies that the number of non-predetermined variables, ν, is equal to $\tilde{\delta} + \max(0, h - \tilde{m})$.

Therefore, Lemma 1 still holds for Rule (7) instead of Rule (5), if δ, m, and $Q(z)$ are respectively replaced by $\tilde{\delta}$, $\tilde{m}$, and $\tilde{Q}(z)$ in this lemma. As a consequence, Propositions 5-10 still hold for Rule (7) instead of Rule (5), if δ, m, and $Q(z)$ are respectively replaced by $\tilde{\delta}$, $\tilde{m}$, and $\tilde{Q}(z)$ in these propositions.

Now suppose that the system composed of Model (4) and Rule (5) is regular (as stated in the proposition), i.e. that $q_{\mathcal{C}} = r_{\mathcal{C}} = 0$. Since $q_{\mathcal{C}} = 0$, we have $\forall j \in \{1, ..., J\}$, $\underline{\phi}_j > 0$. Suppose further that $\sum_{j=1}^{J} |\phi_j| / \underline{\phi}_j < 1$ (as also stated in the proposition). Let $\tilde{q}_{\mathcal{C}} := \#\{z \in \mathcal{C} | \tilde{Q}(z) = 0\}$ denote the number of roots of $\tilde{Q}(z)$ on $\mathcal{C}$ (counting multiplicity). For any $z \in \mathcal{C}$, we have $\tilde{Q}(z) = z^g Q(z)\{1 + \sum_{j=1}^{J} \phi_j [R_j(z)/Q(z)] z^{m_j-h_j}\}$ with $z^g \neq 0$, $Q(z) \neq 0$, and

$$\begin{aligned} \left|1 + \sum\nolimits_{j=1}^{J} \phi_j \frac{R_j(z)}{Q(z)} z^{m_j-h_j}\right| &\geq 1 - \left|\sum\nolimits_{j=1}^{J} \phi_j \frac{R_j(z)}{Q(z)} z^{m_j-h_j}\right| \\ &\geq 1 - \sum\nolimits_{j=1}^{J} |\phi_j| \left|\frac{R_j(z)}{Q(z)}\right| \geq 1 - \sum\nolimits_{j=1}^{J} \frac{|\phi_j|}{\underline{\phi}_j} > 0. \end{aligned}$$

So, we get $\tilde{q}_{\mathcal{C}} = 0$. Since $\tilde{q}_{\mathcal{C}} = r_{\mathcal{C}} = 0$, the system composed of Model (4) and Rule (7) is regular.

Moreover, let $\tilde{q} := \#\{z \in \mathbb{C} | \tilde{Q}(z) = 0, |z| < 1\}$ denote the number of roots of $\tilde{Q}(z)$ inside $\mathcal{C}$ (counting multiplicity). To determine $\tilde{q}$, I apply Rouché's theorem to $\mathcal{J} = \mathcal{C}$, $P_b(z) = z^g Q(z)$, and $P_s(z) = z^g \sum_{j=1}^{J} \phi_j R_j(z) z^{m_j-h_j}$. For any $z \in \mathcal{C}$, we have

$$\begin{aligned} |z^g Q(z)| = |Q(z)| &> |Q(z)| \sum\nolimits_{j=1}^{J} \frac{|\phi_j|}{\underline{\phi}_j} \geq |Q(z)| \sum\nolimits_{j=1}^{J} |\phi_j| \left|\frac{R_j(z)}{Q(z)}\right| \\ &= \sum\nolimits_{j=1}^{J} \left|\phi_j R_j(z) z^{m_j-h_j}\right| \geq \left|\sum\nolimits_{j=1}^{J} \phi_j R_j(z) z^{m_j-h_j}\right| = \left|z^g \sum\nolimits_{j=1}^{J} \phi_j R_j(z) z^{m_j-h_j}\right|. \end{aligned}$$

So, $\tilde{Q}(z)$ has the same number of roots inside $\mathcal{C}$ as $z^g Q(z)$: $\tilde{q} = q + g$. As a consequence,

$\tilde{q}-\tilde{\delta}=q-\delta$, and the determinacy status under Rule (7) with $\phi=0$ is the same as under Rule (5) with $\phi=0$.

Finally, let $\underline{\phi} := \min_{z\in\mathcal{C}} |Q(z)/R(z)|$ and $\bar{\phi} := \max_{z\in\mathcal{C}} |Q(z)/R(z)|$ denote the thresholds for ϕ under Rule (5), and $\underline{\tilde{\phi}} := \min_{z\in\mathcal{C}} \left|\tilde{Q}(z)/R(z)\right|$ and $\bar{\tilde{\phi}} := \max_{z\in\mathcal{C}} \left|\tilde{Q}(z)/R(z)\right|$ the thresholds for ϕ under Rule (7). We have

$$\begin{aligned}\underline{\tilde{\phi}} &= \min_{z\in\mathcal{C}} \left|\frac{Q(z)}{R(z)}\left[1+\sum\nolimits_{j=1}^{J}\phi_j\frac{R_j(z)}{Q(z)}z^{m_j-h_j}\right]\right| \geq \min_{z\in\mathcal{C}}\left|\frac{Q(z)}{R(z)}\right|\min_{z\in\mathcal{C}}\left|1+\sum\nolimits_{j=1}^{J}\phi_j\frac{R_j(z)}{Q(z)}z^{m_j-h_j}\right| \\ &\geq \underline{\phi}\min_{z\in\mathcal{C}}\left[1-\sum\nolimits_{j=1}^{J}|\phi_j|\left|\frac{R_j(z)}{Q(z)}\right|\right] \geq \underline{\phi}\left[1-\sum\nolimits_{j=1}^{J}|\phi_j|\max_{z\in\mathcal{C}}\left|\frac{R_j(z)}{Q(z)}\right|\right] = \underline{\phi}\left(1-\sum\nolimits_{j=1}^{J}\frac{|\phi_j|}{\underline{\phi}_j}\right)\end{aligned}$$

and

$$\begin{aligned}\bar{\tilde{\phi}} &= \max_{z\in\mathcal{C}} \left|\frac{Q(z)}{R(z)}\left[1+\sum\nolimits_{j=1}^{J}\phi_j\frac{R_j(z)}{Q(z)}z^{m_j-h_j}\right]\right| \leq \max_{z\in\mathcal{C}}\left|\frac{Q(z)}{R(z)}\right|\max_{z\in\mathcal{C}}\left|1+\sum\nolimits_{j=1}^{J}\phi_j\frac{R_j(z)}{Q(z)}z^{m_j-h_j}\right| \\ &\leq \bar{\phi}\max_{z\in\mathcal{C}}\left[1+\sum\nolimits_{j=1}^{J}|\phi_j|\left|\frac{R_j(z)}{Q(z)}\right|\right] \leq \bar{\phi}\left[1+\sum\nolimits_{j=1}^{J}|\phi_j|\max_{z\in\mathcal{C}}\left|\frac{R_j(z)}{Q(z)}\right|\right] = \bar{\phi}\left(1+\sum\nolimits_{j=1}^{J}\frac{|\phi_j|}{\underline{\phi}_j}\right).\end{aligned}$$

A.16 Proof of Proposition 12

Under Rule (5) with $\phi \neq 0$, as stated in Lemma 1, we have $\nu = \delta + \max(0, h-m)$ and $P(z) = Q(z)z^{\max(0,h-m)} + \phi R(z)z^{\max(0,m-h)}$. Under Rule (8) with $\phi \neq 0$, the number of non-predetermined variables is still $\nu = \delta + \max(0, h-m)$, since the new terms in the rule are past (as opposed to expected future) values of the policy instrument. Moreover, the characteristic polynomial of the dynamic system is still the same as the characteristic polynomial of the corresponding perfect-foresight system, but the latter system is now

$$\begin{bmatrix} \mathbf{A}(L) & L^{-\gamma}\mathbf{B}(L) \\ -\phi L^{-h}\mathbf{V}(L) & \rho(L) \end{bmatrix}\begin{bmatrix} \mathbf{X}_t \\ i_t \end{bmatrix} = \mathbf{0}.$$

Except possibly for a zero-measure set of values of ϕ, I can use the same standard result in time-series analysis as in Appendix A.6. I get that there exists $k \in \mathbb{Z}$ such that $P(z)$, the reciprocal polynomial of the characteristic polynomial, is

$$P(z) = z^k \det\begin{bmatrix} \mathbf{A}(z) & z^{-\gamma}\mathbf{B}(z) \\ -\phi z^{-h}\mathbf{V}(z) & \rho(z) \end{bmatrix}.$$

Using the Laplace expansion and the notations introduced in the main text, I rewrite $P(z)$ as $P(z) = z^k\{\det[\mathbf{A}(z)]\rho(z) - \phi z^{-\gamma-h}W(z)\} = z^k[Q(z)\rho(z) + \phi z^{m-h}R(z)]$. As a reciprocal polynomial, $P(z)$ is such that $P(0) \neq 0$; moreover, we have $Q(0) \neq 0$, $\rho(0) \neq 0$, and $R(0) \neq 0$; as a consequence, we get $k = \max(0, h-m)$, and thus $P(z) = Q(z)\rho(z)z^{\max(0,h-m)} + \phi R(z)z^{\max(0,m-h)}$. So, Lemma 1 still holds for Rule (8) instead of Rule (5) and for $\phi \neq 0$, if $Q(z)$ is replaced by $Q(z)\rho(z)$ in this lemma. As a consequence, Propositions 5-10 still hold for Rule (8) instead of Rule (5) and for $\phi \neq 0$, if $Q(z)$ is replaced by $Q(z)\rho(z)$ in these propositions.